Generalization of Khovanov homology to tangles

These (unfinished) notes aim to bring together some papers on the Rasmussen invariant using the generalization of the Khovanov homology to tangles following [Bar05]. The setting given by Bar Natan in [Bar02] gave a new approach to Khovanov homology which makes a more categorical and abstract room for it, hence allows to generalize results and to give alternative proofs/definitions.

0.1 Bar Natan's setting for Khovanov and Lee homologies

In [Bar05], Bar Natan generalizes Khovanov homology to tangles. He associates to a tangle (up to isotopy) a complex over $Mat(Cob^3/I)$ where I is a collection of relations between the dotted cobordisms.
Recall that $[[T]]$ lives in $(Kob^3_i)_h$ and using this setting, we also get the invariance of the isomorphism classes of $[[T]]$ up to isotopy, that is if T and T' are isotopic links then there exists an homotopy equivalence between the complexes $[T]$ and $[T']$. At the end of his paper, Bar Natan introduces a new and simpler category which is the dotted cobordism category. Replacing Cob^3 by this latter one, homotopy invariance of the complex up to isotopy does not hold for tangles anymore, but it holds for knots and links and moreover we recover again the Khovanov and Lee theories. This latter category will be our setting.

0.1.1 The Khovanov bracket for tangles

Definition 0.1. *A tangle is a piece of link. A tangle diagram is a piece of a link diagram.*

The idea of generalization of the Khovanov bracket arises naturally from its definition. Remember that we defined the Khovanov bracket transforming the cube of resolutions of a link diagram into a cube where vertices were decorated by closed 1-manifolds and edges by 2-dimensional cobordisms between them. This picture was created only by local considerations of the diagram. We shouldn't have a lot of difficulty to adapt this picture to tangles.

Let T be an oriented tangle diagram. Through the picture below which summarizes the construction of the Khovanov complex, one can apply the same procedure to a tangle diagram to get an n-dimensional cube of resolutions for the tangle.

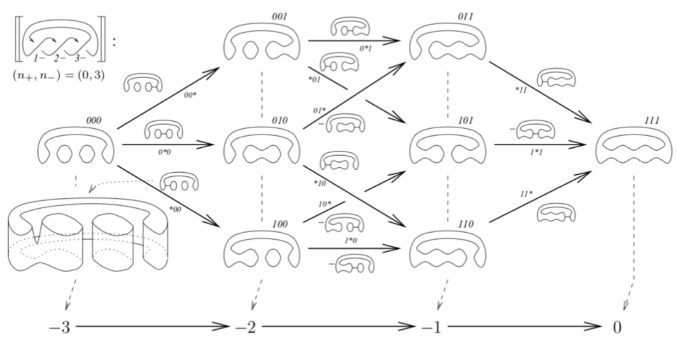

Figure 1

Below we summarize how we form the Khovanov bracket for an oriented link diagram. Attention, this is not the generalization of the Kauffman bracket since we consider oriented diagrams here.
Let T be an oriented tangle diagram with n crossings. Form a complex $[T]$ by:

1) Consider the n-dimensional cube of resolutions

2) Transform the cube by taking for each vertex the corresponding 1-dimensional manifold and for each edge the corresponding cobordism.

3) To get the Khovanov bracket $[T]$, flatten the cube to a formal chain complex where chains are formal direct sums of 1-manifolds and edges are formal matrices with cobordism entries such that the homological grading corresponds to the height of the resolutions shifted by n_-.

If we want to generalize the Khovanov complex to tangles, we should first define an adapted cobordism category. Let $B \in \partial D^2$ be a finite collection of points on the unit disk.

Definition 0.2. *If T, T' are two tangle diagrams having their ending points on B, a cobordism between them is a compact oriented surface $S \subset R^2 \times I$ such that $S \cap \partial(R^2 \times I) = T \times \{0\} \cup B \times I \cup T' \times \{1\}$.*

Definition 0.3. *The category $Cob^3(B)$ is given by*

- *Objects are resolutions of tangle diagrams such that $\partial T = B$*
- *Morphisms are cobordisms between them up to boundary preserving isotopy*

The composition is given by placing one cobordism atop the other

Remember that the Khovanov bracket for links has for chain spaces direct sums of modules and as differentials matrices with entries being linear maps. We would like to make a room for such a setting in a pre-additive category.

Definition 0.4. *Recall that a pre-additive category is a category where every hom-set is an abelian group such that the compositions are bilinear.*
If C is a pre-additive category, define its additive closure $Mat(C)$ by

- *Objects are formal direct sums $\bigoplus_{i=1}^{n} O_i$ of objects in C*
- *Morphisms are matrices having entries the morphisms of C*

We can add morphisms using matrix addition and the composition is given by the matrix multiplication. In particular $Mat(C)$ is still a pre-additive category.

Now the picture defining the usual Khovanov complex for links can be interpreted as a complex over $Mat(Cob^3)$, informally saying we can represent chain spaces by column vectors and differentials by arrows between them.

Definition 0.5. *If C is a pre-additive category, we let $Kom(C)$ to be the pre-additive category given by:*

- *objects are chains of finite length over C*
- *morphisms are complex maps ie. a collection of commutative diagrams.*

The composition is given by chain wise composition ie. $(F \circ G)^r = F^r \circ G^r$.

Definition 0.6. *Let C be a category. We say that two morphisms $F, G : (\Omega_1^r) \to (\Omega_2^r)$ in $Kom(C)$ are homotopic if there exists "backwards diagonal" morphisms $h^r : \Omega_1^r \to \Omega_2^{r-1}$ so that $F^r - G^r = h^{r+1}d^r + d^{r-1}h^r$ for all r*

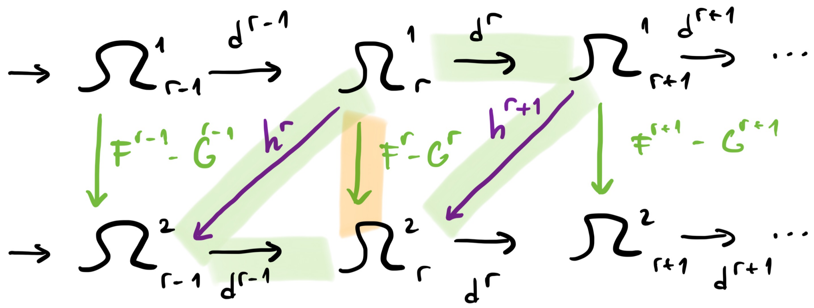

Figure 2

Many of the usual properties of homotopies remain true in the formal case, with essentially same proofs. In particular homotopy is an equivalence relation and is invariant under composition both on the left and the right.

Definition 0.7. *$Kom_h(C)$ is $Kom(C)$ modulo homotopies. That is, its objects are the same as in $Kom(C)$ and the morphisms are homotopy classes of morphisms of $Kom(C)$. As usual, we say that two complexes in $Kom(C)$ are homotopy equivalent if they are isomorphic in $Kom_h(C)$. This is an equivalence relationship on complexes.*

Definition 0.8. *Let $B \in \partial D^2$. We denote*

$$Kob(B) = Kom(Mat(Cob^3(B)))$$

Proposition 0.1. *For a tangle diagram T, $[T]$ is a complex in $Kob(\partial T)$*

0.1.2 Invariance under Reidemeister moves

In this section, we will show that the complex $[T]$ is a homotopy invariant of the tangle T in some *convenient* category to define soon. The main ingredient to define this convenient category is to endow $Mor(Cob^3(B))$ (ie. its morphisms) by some *local relations* and then to consider the quotient category.

Quotient cobordism categories

On $Mor(Cob^3(B))$ we consider the following local relations

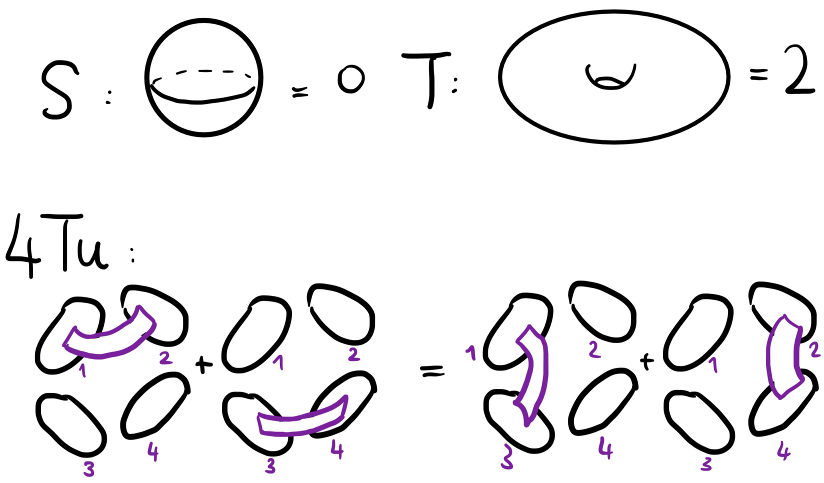

Figure 3

It is not hard to see that the relations are well defined with respect to the composition relations and that the quotient category $Cob^3(B)/l$ is a pre-additive category. Also reload the notation $Kob_h = Kom_h(Cob^3(B)/l)$.

Theorem 0.1. *The isomorphism class of $[T] \in Kob_h$ is an invariant of the tangle T.*

Here we show the invariance under the Reidemeister move 1 , and our proof for now is local. The idea for the other Reidemeister moves is similar. In later sections we will see how our local proof will generalize to the entire tangle.

Proof. **Reidemeister 1 move:** Our goal is to show that the complexes below

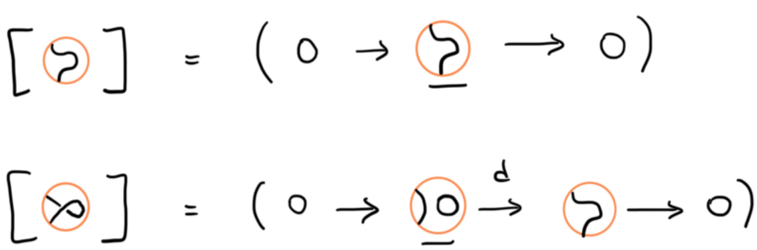

Figure 4

are homotopy equivalent where

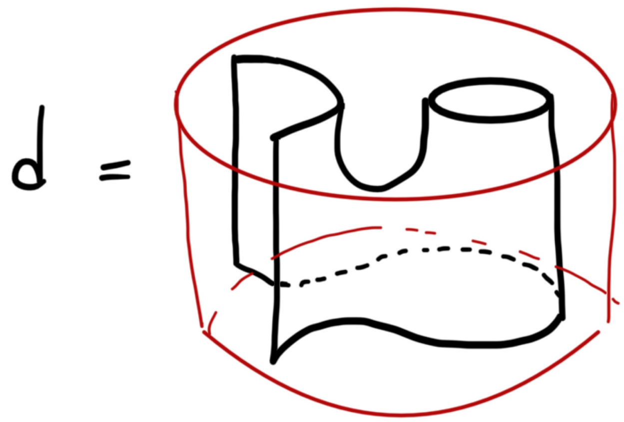

Figure 5

. To do so, we construct homotopically inverse morphisms

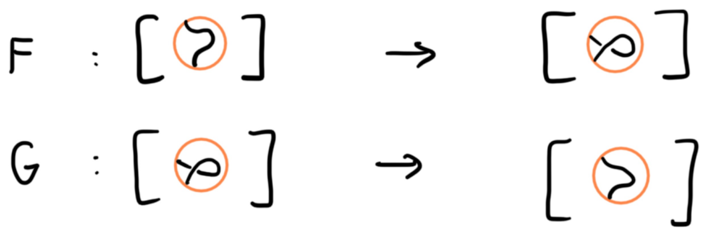

Figure 6

We have that $F^i = G^i = 0$ for $i \neq 0$ and define F^0, G^0 by

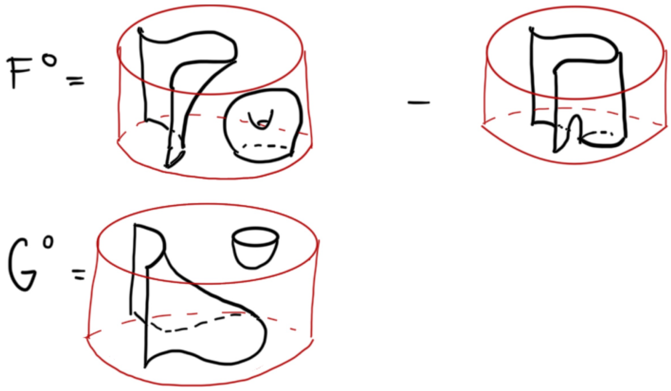

Figure 7

Then $G \circ F = I$ is given by the T relation

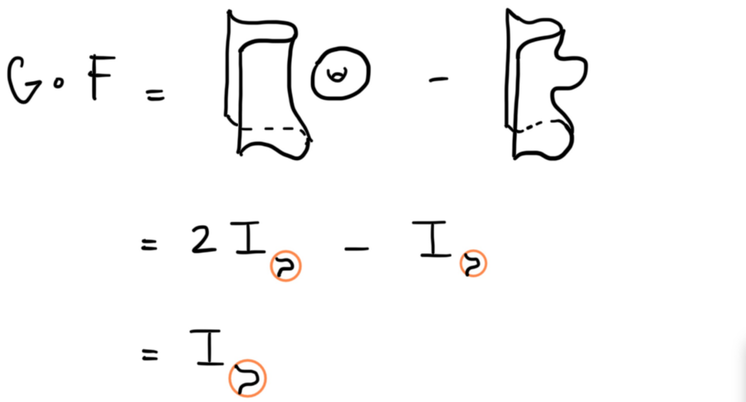

Figure 8

Now consider the homotopy morphism h given by

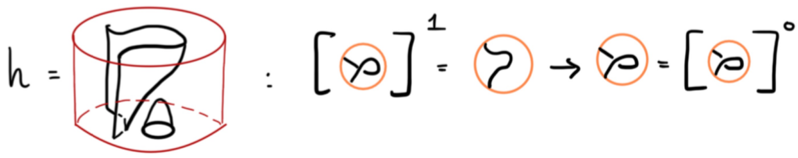

Figure 9

Then we have $F^1 \circ G^1 - I + dh = -I + dh$ Moreover by the $4Tu$ relation we have $F^0 \circ G^0 - I + hd = 0$. To see this, consider the cobordism C below

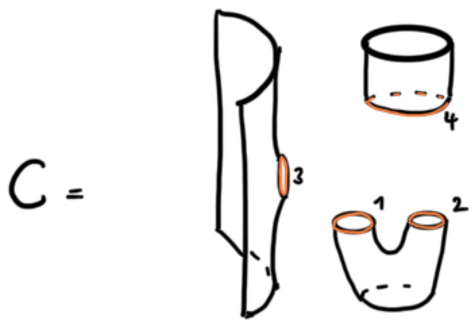

Figure 10

Then applying the $4Tu$ relation one gets

$$C_{12} + C_{34} = C_{13} + C_{24}$$

Where

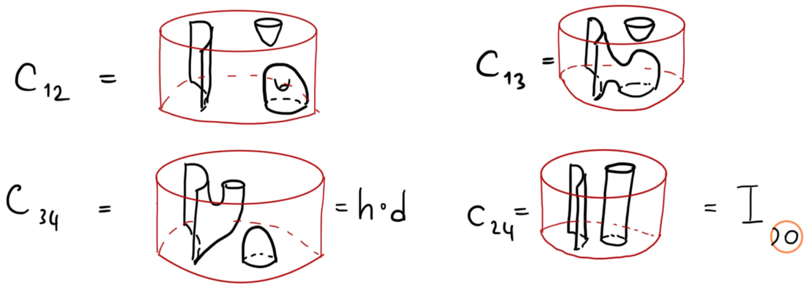

Figure 11

Hence $F^1 \circ G^1 - I = -hd$

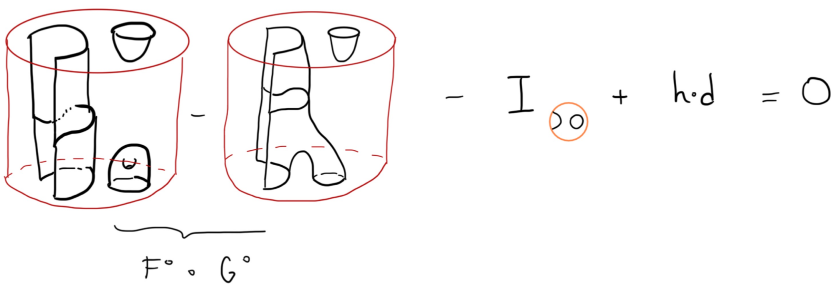

Figure 12

$\square$

Planar algebras and tangle compositions

Below is a formal definition of a tangle.

Definition 0.9. *A tangle is an embedding of n arcs and m circles into $R^2 \times [0,1]$. A (planar) tangle diagram is a collection of curves in a planar disk which can be either* *closed either having their beginning/ending points on the boundary circle. Hence ∂T is a collection of points on the boundary circle.*

Definition 0.10. *A smoothing of a tangle is a collection of circles in the interior of the disk and arcs having their beginning/ending points on the boundary circle.*

Tangles can be composed in a variety of ways. Indeed, any d-input "planar arc diagram" D yields an operator and takes d tangles as inputs and producing a single "bigger" tangle as an output by placing the d inputs into the d holes of D.
The purpose of this section is to define precisely what planar arc diagrams are and explain how they turn the collection of tangles into a planar algebra, to explain how formal complexes in Kob also form a planar algebra and to note that the Khovanov bracket [] is a planar algebra morphism from the planar algebra of tangles to the planar algebra of such complexes. Thus, Khovanov bracket "compose well". In particular our local proof for the Reidemeister invariance lifts to the global invariance under Reidemeister moves.

Definition 0.11. *A planar arc diagram $D$is a "big" output disk with k points on the boundary circle and d ordered "smaller" input disks with k_i points on their boundaries for $0 \leq i \leq d$, and oriented arcs which are in the complement of the interiors of the input disks, having their endings/beginning on the points on the boundaries of the input/output disks. Moreover the output and each input disk is endowed with a base point. An unoriented planar arc diagram is obtained by forgetting the orientations of the arcs.*

Definition 0.12. *If B is a collection of points on the unit disk, we denote $\mathcal{T}^0(k)$ the collection of unoriented tangles T in a based disk such that $\partial T = B$. If s is an n-tuple of $\uparrow$ and $\downarrow$ with $|s| := n$, we denote $\mathcal{T}^0(s)$ the collection of all (oriented) tangles in a based disk having their endpoints prescribed by s. We denote by $\mathcal{T}(k), \mathcal{T}_0(s)$ the quotients by the three Reidemeister moves.*

Example 0.1. *Here is an example of an oriented arc diagram for $s = \uparrow\uparrow\downarrow\downarrow\downarrow$*

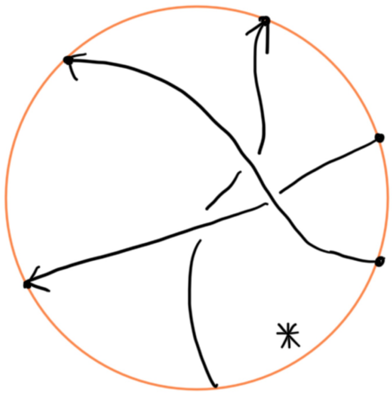

Figure 13

A planar arc diagram can be viewed naturally as an operator on tangles having the good number of ending points.

Definition 0.13. *Let Dbe a planar arc diagram with k endings and d "input disks, each having k_i prescribed endings with $i \in [[1, d]]$. Then Dinduces an operator c*

$$D : \mathcal{T}(k_1) \otimes \cdots \otimes \mathcal{T}(k_d) \to \mathcal{T}(k)$$

by placing the d input diagrams into the d holes of D. Likewise if D is oriented, D defines an operation

$$D : \mathcal{T}(s_1) \otimes \cdots \otimes \mathcal{T}(s_d) \to \mathcal{T}(s)$$

Example 0.2. *Here is an example of a planar arc diagram and how it induces an operation on the concerned tangles*

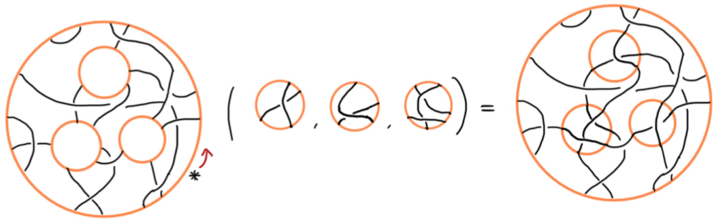

Figure 14

Example 0.3. *A natural example of a planar arc diagram is the one input disk having a single input disk and k arcs connecting k points on the two boundary circles. Hence D induces the identity operation on the set $T(k)$ of tangles with k endings. Moreover these operations are compatible with each other "associative" in the following way:*

$$D_i = D \circ (I \times \cdots \times D' \times \cdots \times I)$$

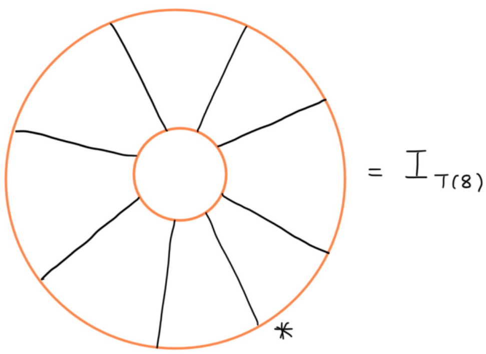

Figure 15

Definition 0.14. *A planar algebra is a collection $\{P(B)\}_B$ of sets indexed by sets of boundary points along with operations D defined for each d-input planar arc diagram* *satisfying the identity rule*

$$I_B = id_{P(B)}$$

an the associativity rule.

Definition 0.15. *A planar algebra morphism between two planar algebras is a collection of maps* $\{\phi_B\}, \phi_B : P^a(B) \to P^b(B)$ *verifying*

$$D \circ (\phi_{B_1} \times \cdots \phi_{B_d}) = \phi_B \circ D$$

for an operation induced by a planar arc diagram with $\partial D = B$.

Remark 0.1. *Hence applying the morphism to d tangles and applying a planar arc diagram should be the same as applying D and then taking the image by the morphism. In other words, planar morphisms allow us to work locally!*

Example 0.4. *Two natural examples that we have been secretly thinking about while defining the Khovanov complex for tangles were* $Obj(Cob^3_\bullet(B))$ *and* $Mor(Cob^3_\bullet(B))$*. One can see* $Obj(Cob^3_\bullet(B))$ *as a sub-algebra of* $(T(B))_B$
$Mor(Cob^3_\bullet(B))$ *are dotted cobordisms, and any arc diagram D in* $Obj(Cob^3_\bullet(B))$ *can be turned into an operator by taking* $D \times I$ *which is the product cobordism outside the input disks together with cylinder holes between the input disks as illustrated below.*

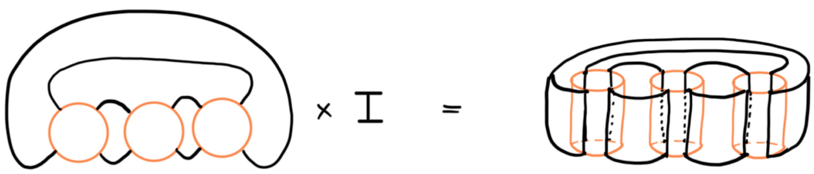

Figure 16

Example 0.5. *Planar arc diagrams allow us to have "tensor product" operations on the collection of* $Obj(Cob^3_\bullet(B))'s$ *and that of* $Mor(Cob^3_\bullet(B))'s$*. Indeed the "tensor product" between d smoothings is given by the induced operator on corresponding planar algebras* *by the planar arc diagram D with k endings being B and having d input disks each with k_i endings such that $\sum_i k_i = k$*

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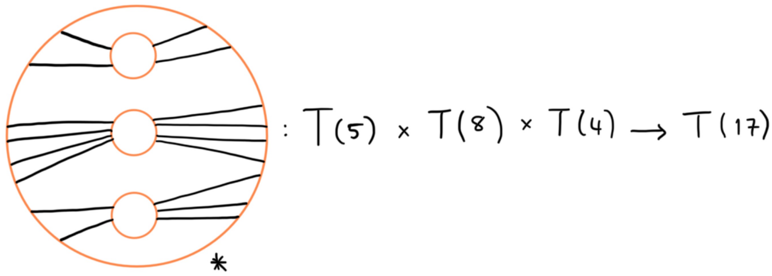


Figure 17

We will sometimes call these tensor product inducing arc diagrams shortly the tensor product or in the case of tangles, "horizontal composition".

Example 0.6. *This last example shows how we actually get a planar algebra stucture on $Kob(B)$ ie. over complexes over $Mat(Cob^3/I_o(B))$. Indeed, the operations will correspond to tensor product operation. Let D be a tensor arc diagram, then we can extend the corresponding operations induced by D to $Mat(Cob^3/I_o(B))$. Let Ω_1, Ω_2 be two complexes over $Mat(Cob^3/I_o(B))$. Define the complex $(\Omega_1 \times \Omega_2, d)$ over $Mat(Cob^3/I_o(B))$ by:*

$$\Omega_r := \bigoplus_{r_1+r_2=r} D(\Omega_1^{r_1}, \Omega_2^{r_2})$$
$$d_{|D(\Omega_1,\Omega_2)} := D(d_1, I_{\Omega_2}) + (-1)^{r_2} D(I_{\Omega_1}, d_2)$$

Lemma 0.1. *Homotopies at the level of tensor factors induce homotopies at the level of tensor products.*

Here is the most important result which allows us to work locally while working in Khovanov homology.

Theorem 0.2. *(1) $Kom(B)$ has a natural structure of planar algebra.*

(2) The morphism D in $Kob(B)$ sends homotopy equivalent chains to homotopy equivalent chains, hence $Kob_h(B)$ is also a planar algebra.

(3) The bracket $[.]$ *descends to an oriented planar algebra morphism*

$$[.] : \mathcal{T}(s) \rightarrow Kob_h(s)$$

In particular this completes the proof of the invariance of the complex $[T]$ *in* $Kob_h(\delta T)$.

Proof. Let T be a tangle with n crossings X_i. Without loss of generality denote by $[[.]]$ any of the two brackets. Indeed we have by definition

$$[[T]] = [[D(X_1, \cdots, X_n]]$$

where D is the tensor product inducing arc diagram. Now by the definition of the tensor product induced by D on complexes and that of $[[.]]$, we have

$$[[D(X_1, \cdots, X_n]] = D([[X_1]], \cdots, [[X_n]])$$

Hence

$$[[T]] = \bigotimes [[X_1]] \cdots [[X_n]]$$

$\square$

Hence, we retain that a tensor planar diagram D induces a tensor product on complexes over $Mat(Cob^3_{\bullet})$ in the defined way and this implies that the Khovanov bracket is a planar algebra morphisms. This is the main point allowing to generalize proofs concerning a single crossing to n crossing tangles without induction, since $[[.]]_i$ are planar algebra morphisms.

Gradings and a minor refinement

In this paragraph we introduce gradings which will lead to a refinement of the theorem of invariance. The proof has no additional difficulty, but the benefits are great, as we saw in the case of links, it will generalize the Khovanov complex via the Khovanov bracket and relate it to the unnormalized Jones polynomial.

Definition 0.16. *A graded category is an additive category such that*

- *the homomorphism groups* $Hom(O_1, O_2)$ *are graded abelian groups*
- *the composition maps respect the gradings:* $gr(f \circ g) = gr(f) + gr(g)$
- *identity maps are of degree 0*
- *There is a* $\mathbb{Z}$*-action* $(m, O) \mapsto O\{m\}$ *on the objects of* C.

Remark 0.2. *Observe that as plain abelian groups we have*

$$Mor(0_1, O_2) = Mor(0_1\{m_1\}, O_2\{m_2\})$$

but the gradings do change. If $f \in Mor(0_1, O_2)$ *has grading* d *then* $f \in Mor(0_1\{m_1\}, O_2\{m_2\})$ *has grading* $d - m_1 + m_2$.

If C is a graded category, then $Mat(C)$ can also be considered as a graded category. A morphism is homogeneous of degree d if all of its entries are homogeneous of degree d. Then in a similar way, the complexes in $Kom(C)$ become graded aswell.

We introduce a quantum grading on our cobordism category

Definition 0.17. *Let* $C \in Mor(Cob^3(B))$ *be a cobordism. Define*

$$deg(C) = \chi(C) - \frac{1}{2}|B|$$

Remark 0.3. *In particular the degree is additive under the vertical composition but also in horizontal composition. Moreover this grading coincides with our definition of the quantum grading for the links since we take* $B = \emptyset$.

Remark 0.4. *The relations given by* I_0 *are degree homogeneous.*

By this remark, we get that $Cob^3/I_0(B)$ is a graded category, so are the categories $Kob(B)$ and $Kob_h(B)$.

Defining the Khovanov complex for tangles

Let T be a tangle diagram with $n_+(T), n_-(T)$ positive and negative crossings.

Definition 0.18. *Let* $Kh(T)$ *be the chain complex whose chain spaces are given by*

$$CKh^r(T) := [T]\{r + n_+ - n_-\}$$

Hence we get a refinement for our first theorem

Theorem 0.3. *(1) All differentials in* $CKh(T)$ *are of degree 0*

(2) $CKh(T)$ *is an invariant of the tangle* T *up to degree 0 homotopy equivalence*

(3) CKh *descends to an oriented planar algebra morphism and all the planar algebra operations are of degree* 0.

Proof. The first assertion follows from the fact that the saddle cobordism is of degree -1, and between two vertices r increases by 1. For the second, see details of the invariance for the Khovanov bracket. For the third assertion, it follows from the corresponding proof for the Khovanov bracket and the fact that n_+ and n_- are additive under planar algebra operations. □

0.1.3 Applying a TQFT to get a homology theory

Now it is time to point out a remark about the generalization of the Khovanov complex to tangles. As we see, we obtained the invariance of a complex whose chain maps are not $\mathbb{Z}$-modules. But the Khovanov homology and the Khovanov complex for the case of links was defined over $\mathbb{Z}$-modules via a $(1+1)$ TQFT. Indeed, even though the Khovanov complex in this chapter has excellent composition properties, the target space $Kob(B)$ seems quite unmanageable since given two formal complexes, it is quite hard to decide if two they are homotopy equivalent. Hence, we will apply a TQFT and obtain a homology theory. We will lose some information along the way ut we will get a computable tangle invariant.

Recovering the Khovanov homology for link

First, let's present the general setting to see how one obtains homology theories for tangles. Let $\mathfrak{A}$ be an abelian category, for instance the category of vector spaces or $\mathbb{Z}$-modules for example. A functor

$$F : Cob^3(B)/I_0 \to \mathfrak{A}$$

extends right away on $Mat(Cob^3(B)/I_0)$ by taking formal direct sums into the honest direct sums. Hence, for any tangle diagram T, $F(CKh(T))$ is an ordinary complex over an abelian category and applying the functor F to all homotopies, we see that $F(CKh(T))$ is an up to homotopy invariant of the tangle T, so the isomorphism class of $H^*(F(CKh(T)))$ is a tangle invariant.
In addition if $\mathfrak{A}$ is graded and F is degree preserving then the homology $H^*(F(CKh(T)))$ is a graded invariant of T.

If the functor F is only partially defined, for example say only on $Cob^3/I_0(\emptyset)$, then we get a partially defined homological invariant.
For example, let's take $B = \emptyset$ ie. links and $F = \mathcal{A}$ which is our $(1+1)$ TQFT to define the Khovanov homology in Chapter 2. This yields exactly to our construction of the original Khovanov complex for links in Chapter 2.

Proposition 0.2. *$\mathcal{A}$ descends to a functor*

$$\mathcal{A} : Cob^3/I_0(\emptyset) \to \mathbb{Z}Mod$$

Proof. We check the l relations:

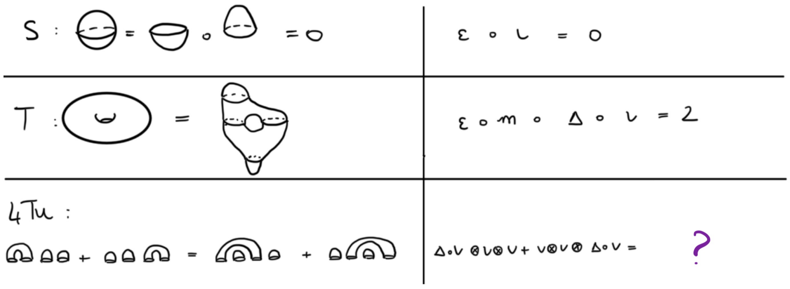


Figure 18

□

A quick comparison of definitions now show that indeed $H^*(\mathcal{A}(CKh(\emptyset)))$ is equal to the original Khovanov homology for links that we defined in chapter 2.

0.2 More functors on Cob^3/I, Khovanov and Lee theories

We have seen how we obtain a more computable invariant via a functor F. In particular, one can wonder what other funtors F one can consider to recover Khovanov homology and also Lee homology. We seek to construct many such functors. Right below, we start with a systematic construction of *tautological* functors which will lead back to the original Khovanov homology, Lee homology and to a new variant defined by Bar Natan over the field with two elements $\mathbb{F}_2$.

Definition 0.19. *Let $B \in \partial D^2$ and fix an object O in $Cob^3/l(B)$. If $B = \emptyset$, fix O to be the empty smoothing. The tautological functor $F_1 : Cob^3/l(B) \to \mathbb{Z}Mod$ is defined on objects by $F_O(O') = Mor(O, O')$ and on morphisms by composition $F_O(f)(g) = f \circ g$.*

Since the morphisms in $Cob^3/I(B)$ appears to be difficult to study, we will often compose these tautological functors with some extra functors that forget some information.

Cutting neck and Khovanov homologies

As a first example, consider $B = \emptyset$ and $O = \emptyset$. Forget all 2-tensors by tensoring with some ground ring where 2 is invertible and mod out by surfaces with $g \geq 2$:

$$F_1(O') = \mathbb{Z}_2 \otimes_{\mathbb{Z}} Mor(\emptyset, O')/(g \geq 2 = 0)$$

Take the cobordism C given by the disjoint union of disks and apply the $4Tu$ relation to them to get the so said *neck cutting relation*

Figure 19

In particular if 2 is invertible this neck cutting relation tells that we can cut open any tube inside a cobordism, replacing it by handles that are localized on one side of the tube.
Repeatedly, cutting tubes in this manner, we see that $\mathbb{Z}_2 \otimes_{\mathbb{Z}} Mor(\emptyset, O')/$ is generated by cobordisms in which every component touches a boundary curve.
Furthermore, reducing with S, T relations and $(g \geq 2 = 0)$ we get a cobordism in which every component has precisely one boundary curve and is either of genus 0 or 1.
Hence if O' is made of k disjoint circles, then

$$F_1(O') = A^{\otimes k}$$

where A is the $\mathbb{Z}_(2)$ where A is the $\mathbb{Z}_{(2)}$-module generated by the two elements $1, X$ defined as below

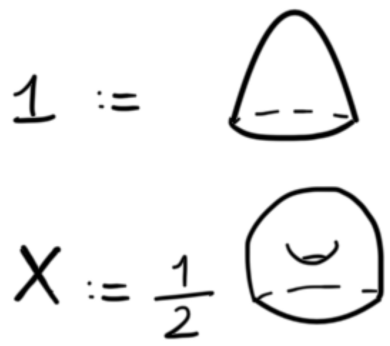


Figure 20

. Indeed we can verify within this basis, F_1 is the same as the TQFT $\mathcal{A}$ and so we recover our original Khovanov homology theory.
Below is the verification. We give the corresponding maps $m_1, \Delta_1, \iota_1, \varepsilon_1$ using the hint in Bar Natan's paper using the bracket notation of quantum mechanics.

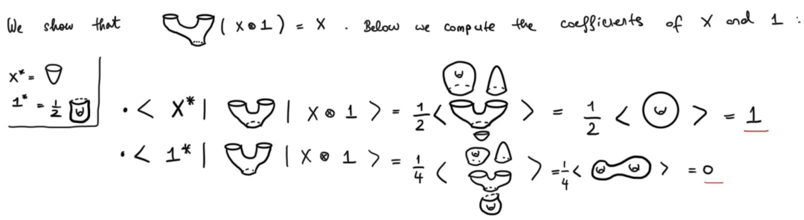


Figure 21

Lee homology and genus 3

At first glance, one may think that the relation $(g \geq 2 = 0)$ is unnecessary in the above discussion. Every high genus surface contains several necks and we can cut those using the neck cut relation to get lower genus surfaces. This clearly work if the genus is high enough, namely if $g \geq 4$:

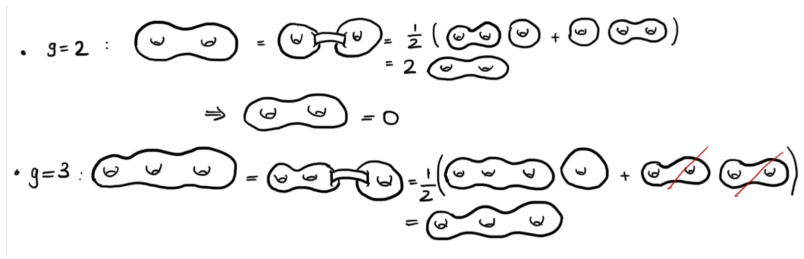


Figure 22

We define the tautological functor F_2 by

$$F_2(O') = \mathbb{Z}_2 \otimes_{\mathbb{Z}} Mor(\emptyset, O')/S_3 = 8$$

As a $\mathbb{Z}_2$-module, $F_2(O')$ is as in the previous example but we lose the gradings and $Delta_2$ and m_2 are modified so that it coincides with the TQFT $\mathcal{A}'$ that we used to get Lee theory. Below are the verifications:

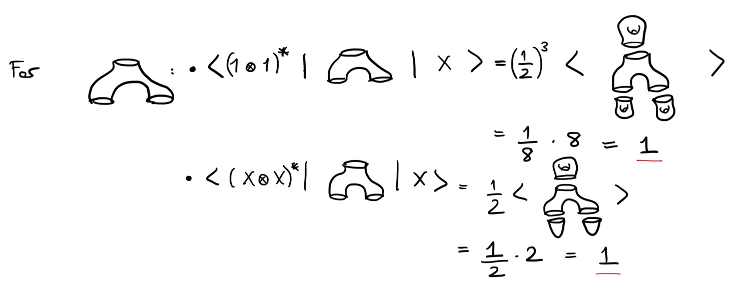


Figure 23

0.2.1 The dotted cobordism category $Cob^3_\bullet$

Another way of recovering Khovanov and Lee theory will be by considering the category of dotted cobordisms $Cob^3_\bullet(B)$. This is extending the category $Cob^3(B)$ by endowing it with a graded variable " $\bullet$ "b such that $b^2 = 0$ and $deg(b) = -2$. In this category a cobordism C between two tangles with k endings can have a finite number of dots which can be placed freely on its connected components, and we define the quantum grading of C by

$$gr(C) := \chi(C) - \frac{1}{2}k - 2n$$

On $Cob^3_\bullet(B)$ consider the following local relations leading to Khovanov complex l_0

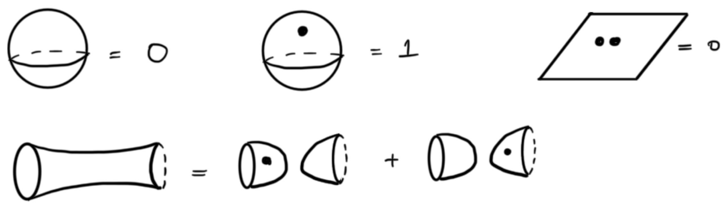


Figure 24

and l_1

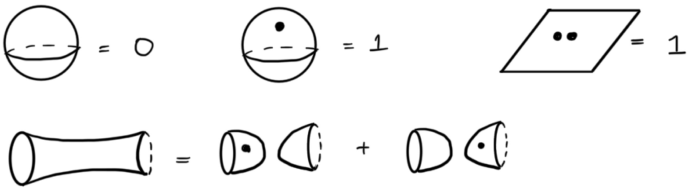


Figure 25

which will lead to the Lee complex Now we can repeat everything with these quotient categories, and the theorem of invariance of the complexes still holds. We call those complexes Khovanov complex denoted $[T]_0$ and Lee complex denoted $[T]_1$ in $Kob_{\bullet,i}(\partial T)$ where $i \in \{0,1\}$ precises which local relation we use. Moreover we have a stronger neck cutting relation and the advantage is that regardless of the ground ring, tubes and handles can be removed. Hence any cobordism will be constructed by a disk or a singly dotted disk and these can be identified with $1, X$ of the TQFT $\mathcal{A}$

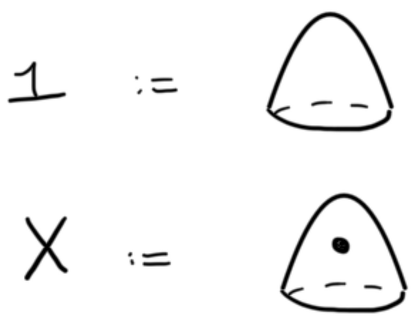


Figure 26

We verify below that these local relations imply the $S, T, 4Tu$ relations, hence pretty much everything we said so far can be repeated working over this category

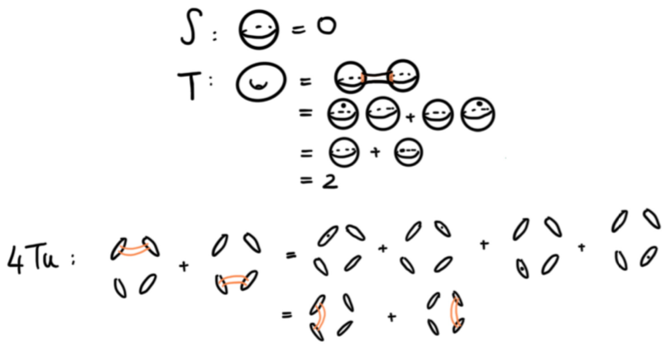


Figure 27

Applying the tautological functors F_1, F_2 we get respectively the Khovanov and Lee homology theories for links without any restriction on the ground ring.

0.3 Lee's structure theorem

In this section we restate and prove, using the dotted cobordism category, a generalized version for tangles of Lee's structure theorem. We work over Bar Natan's dotted cobordism category and we follow Bar-Natan and Morrison's paper [BM06].

Theorem 0.4. *Let T be a n-component tangle. Within the appropriate category the Khovanov-Lee complex of T is homotopy equivalent to a complex with one generator for each orientation of T and with only vanishing differentials.*

0.3.1 The Karoubi envelope

We know from linear algebra that projections are useful tools: they allow to decompose a space into a direct sum of possibly simpler spaces. In any category, the notion of projection still makes sense but the image of a projection may not always make sense. To guarantee this, we consider the *Karoubi envelope* of an additive category.

Definition 0.20. *Let C be an additive category. A projection in C is a morphism $p : O \to O$ such that $p \circ p = p$.*

Definition 0.21. *Let C be an additive category. Then its Karoubi envelope $Kar(C)$ is given by:*

- *Objects are pairs (O, p) with $O \in Obj(C)$ and $p : O \to O$ projection.*

- *Morphisms are triples $(p_1, f, p_2) : (O_1, p_1) \to (O_2, p_2)$ such that $f \circ p_1 = f = p_2 \circ f$*

An object (O, p) in $Kar(C)$ may also be denoted by $im(p)$.

Remark 0.5. *Remark that in $Kar(C)$, the identity morphism of an object (O, p) is (p, p, p) and not (p, id, p) ! Also remark that a projection in C is still a projection in $Kar(C)$.*

Proposition 0.3. *Let C be an additive category where direct sums of objects make sense and let $p : O \to O$ a projection. Then*

- $1 - p : O \to O$ *is a projection*
- $O \cong im(p) \oplus im(1 - p)$ *in $Kar(C)$.*

Proof. Since in an additive category it makes sense to add objects, for the first point it is clear that $(1 - p)^2 = 1 - 2p + p^2 = 1 - p$. For the second point, take the isomorphism $O \to im(p) \oplus im(1 - p)$ given by the matrix $(p \ 1 - p)$ which is clearly invertible with inverse $\begin{pmatrix} p \\ 1 - p \end{pmatrix}$ □

Notice that even though we did add new objects to obtain $Kar(C)$, we don't have any extra morphism. This implies the next important fact, saying that we don't need to verify more homotopy equivalences if we want to work over complexes in $Kar(C)$.

Proposition 0.4. *Let A be an additive category and let C_1 and C_2 be chain complexes in $Kom(C)$. If they are homotopy equivalent in $Kom(Kar(C))$, they were already homotopy equivalent in $Kom(C)$.*

Proof. Let C_i be a complexes over $Kar(C)$. Since a homotopy equivalence is defined by collections of morphisms between the objects of the complexes; two chain maps going in opposite directions and a couple of homotopy maps which are in particular between objects of C, and since we haven't extra morphisms in the Karoubi envelope, we don't have new homotopy equivalences. Hence the homotopy equivalence is in C. □

0.3.2 Red and green projections in Lee theory

Viewed in our perspective, the key ingredient to Lee's theorem is in $Cob^3_{\bullet 1}$, we have the existence of two special projections which we will call r and g for red and green. Indeed, since in Lee theory the local relation on the doubly dotted disk is given by

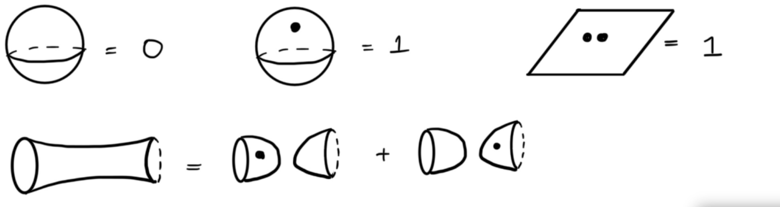


Figure 28

Consider one strand as a smoothing and hence the dotted identity cobordism b verifies $b^2 = 1$.

We define

$$r := \frac{1+b}{2}$$

$$g := \frac{1-b}{2}$$

In particular $r + g = 1$.

Lemma 0.2. *r and g as given above are eigenprojections, moreover they verify following properties*

- *r and g are projections*
- *r and g are complementary i.e. $r + g = 1$*
- *r and g are disjoint i.e. $r \circ g = 0$ and $(strand, r) \circ (strand, g) = (strand \sqcup strand, r \otimes g) = (strand \sqcup strand, 0)$*
- *r and g are eigen-projections of b that is $b \circ r = r$ and $bg = -g$*

Definition 0.22. *A confluence within a smoothing S of a tangle T is a pair of arc segments in S that corresponds to a small neighbourhood of a crossing T. A tangle T with n crossings has n confluences.*

Definition 0.23. *An alternated colouring of a smoothing S of a tangle T is a colouring of the components of S such that two arcs in any confluence have different colourings; red and green.*

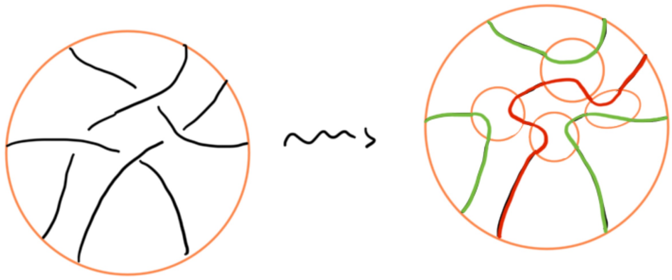


Figure 29

Here is the idea: Since we can place the dot wherever we want on the connected cobordisms, we can place it in particular on the sides of the identity morphisms, ie. on the vertical curtains. The linear relation between r, g, id_{strand} and the fact that r, g are projections allows us to decompose the object $(strand, Id)$ as $im(r) + im(g)$ i.e. as $(strand, r) \oplus (strand, g)$ as below where we draw the object $(strand, r)$ as a red strand and similarly for g.

In this case a 0-smoothing or a 1-smoothing can be viewed as a tensor product of two $(strands, id)$ and hence will be a direct sum of 4 coloured objects as illustrated:

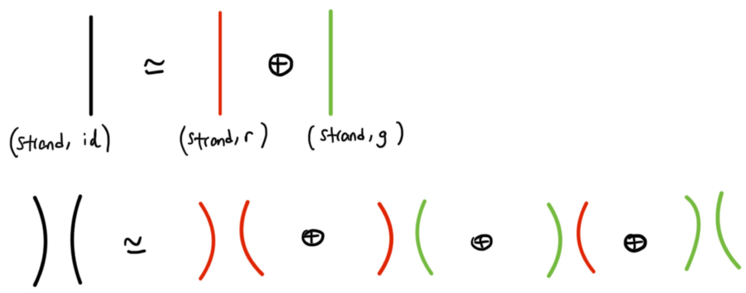


Figure 30

Now for a knot, a dramatic simplification, as says Bar-Natan and Morrison, will occur. Consider the 2-step complex corresponding to a single crossing formed by 2 non vanishing objects and a single differential:

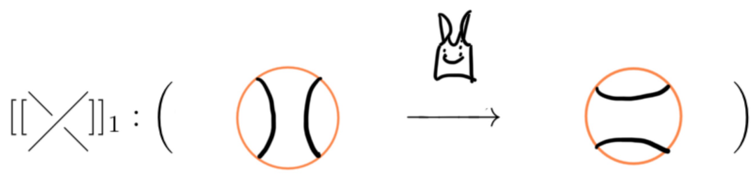


Figure 31

Now by the previous discussion the two objects in the corresponding complex over $Kar(Cob^3_{\bullet 1})$ becomes direct sums of 4 objects and hence the differential is a 4×4 matrix M

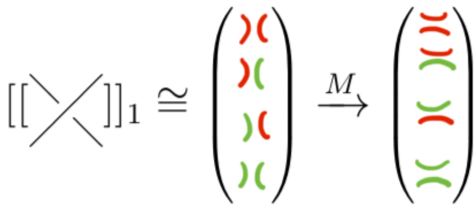


Figure 32

Where the matrix M is given by

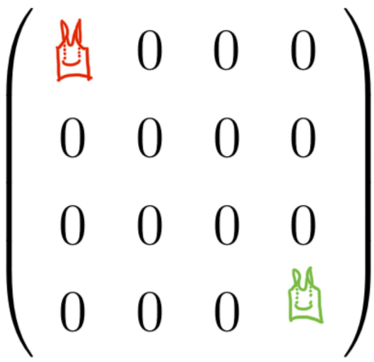


Figure 33

Hence this last complexe is a direct sum of four complexes: The first and the last of the 4 summands being:

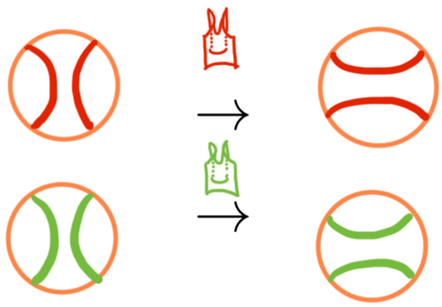
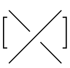


Figure 34

One can show that the red and the green saddle morphisms are invertible if 2 is invertible, their inverses are given by ??. Hence the complexes are contractible ie. their homologies are 0. Hence up to homotopy we get

Figure 35

Hence we found a representative of [[]]$_1$ with a zero differential!

0.3.3 Proving Lee's theorem

We can finally prove Lee's theorem putting all these pieces together. The discussion above shows us that the theorem holds for tangles with a single crossing.

Let T be a tangle with n crossings X_i, where each X_i is a disk containing the crossing. Let D be the planar arc diagram obtained by the tangle T deleting n disk neighbourhoods of crossings. Using the induced operator and using that $[[.]]_1$ is a planar algebra morphism, we get

$$[[T]]_1 = \bigotimes_i [[X_i]].$$

Recall that each complex $[[X_i]]$ has an object for each of the 4 alternately coloured smoothings of X_i and zero differentials from the previous discussion.

From what we defined as tensor product on $Kom(Kar(Mat(Cob^3_\bullet)))$, the differentials of $[[T]]_1$ are all zero. At first sight, it might seem like $[[T]]_1$ has too many objects. Let's zoom on the the verity:
Notice that every alternately coloured smoothing of T appears as an object in $[[T]]$: indeed an alternately coloured smoothing of $[[T]]$ can be divided into a collection of alternately coloured smoothing of such crossing X_i.
Now there also seems to be some extra objects in the tensor product. Those are the ones where the connected arcs have different colorings due to the colourings of the arcs living in different crossings but which connect via an arc. But indeed, in $Kar((Mat(Cob^3_\bullet)))$ these objects are the zero objects since the lemma ??? is true for the horizontal composition aswell. Hence, the extra objects all disappear and we have the good number of objects in the complex!

0.3.4 Alternately coloured resolutions and orientations

Here, we prove that there is a bijection between the orientations of the tangle and its alternately coloured resolutions.

Proposition 0.5. *Let T be an oriented tangle. There is a bijection between the alternately coloured resolutions and tangle orientations.*

Proof. Consider a tangle diagram T and fix a checkboard colouring.
Let $\wr$ be an orientation of the tangle, and take the corresponding oriented resolution which is defined in the same way as for links ie. orienting the resolved arcs consistently. To this resolution we want to associate a colouring of arcs such that two arcs in a confluence have different colours. A way of doing this is considering a confluence and putting a dot on the left of the two arcs, then to colour the arc in the same colour as the region where the dot on the left of the arc lives.
By the lemma (that two strands having the same orientation have different labels) it is clear that the two arcs will have different colours if they live on the same confluence.

Figure 36

Conversely to an alternated smoothing corresponds an orientation. Take such a confluence and give the two strands the parallel orientations so that the checkboard colouring of the regions are on the left of the arcs. We need to check that these local orientations induce an orientation on the whole tangle. Indeed, notice that by the checkboard colouring of the link any two connected confluences will have their connected arcs of the same colour, hence the global orientation remains coherent. Now reproduce the oriented tangle diagram using the smoothing information. It is now clear that the given maps are inverses of each other as illustrated below:

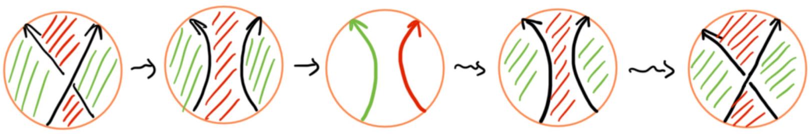


Figure 37

□

Corollary 0.1. *Let L be a link with n components. We get Lee's result :*

$$dim([[L]]_1) = 2^n$$

.

Proof. For a link with n components we have 2^n orientations hence by the proposition, 2^n alternately coloured smoothings. Since the differentials are all zero, this gives the dimension of homology. □

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