Université Paris Cité

Khovanov homology and the Rasmussen invariant

Master's Thesis

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Abstract

Short after the definition of the Khovanov homology at the very end of the $20th$ century by Mikhail Khovanov ([Kho00]), Jacob Rasmussen defined ([Ras10a]) a knot invariant using a deformation of Khovanov homology, namely Lee-Khovanov homology ([Lee02]). The Milnor conjecture concerning the slice genus of torus knots, namely $g_*(T_{p,q}) = \frac{(p-1)(q-1)}{2}$ which was originally proved using gauge theory by Kronheimer and Mrowka ([KM93]) was reproved using the Rasmussen invariant. This purely combinatorial proof is considered to be the first application of Khovanov homology.
Rasmussen invariant s is a knot concordance invariant and it is a group homomorphism from the concordance group to $\mathbb{Z}$, just as the signature. One important reason that s is interesting is that it gives a lower bound for the smooth slice genus of a knot by $s(K) \leq 2g_*(K)$. In particular, it detects knots which are topologically slice but not smoothly slice and hence encodes important geometric information about the knot. It is defined combinatorically so is algorithmically computable and efficient algorithms have been developped [Sch21], however many interesting case remains beyond the reach of current implementations and we do not have in general a formula to compute it except for some families of knots such as positive knots and alternative knots as we shall see.

This master's project is organized as follows: In Chapter 1 we give preliminaries on knots, links and algebraic tools which we will need to define the Rasmussen invariant. In Chapter 2 we define the Khovanov homology, Lee-Khovanov homology and the Rasmussen invariant. At the end of the chapter we prove the Milnor conjecture. In Chapter 3, we mention briefly and without introducing the extra material they contain, several works which contain computations of Rasmussen invariant in different situations.

Acknowledgements

I want to thank my supervisor Professor Marco firstly for having offered to me the chance to discover Khovanov homology and the Rasmussen invariant via this Master's project, but also for making time for a Zoom meeting nearly every week for 3 months and for being friendly and patient with my confusions. During this relatively short time I had the opportunity to learn about many fundamental concept and to get introduced to knot theory via such a beautiful homology theory. Most importantly, I had lot of fun doing this Master's project and learning about the Rasmussen invariant.

Contents

Chapter 1

Preliminairies

This preliminary chapter is divided into two parts. We first give some basic definitions on knots and links mostly following [Mur96], introduce some classical and useful invariants and investigate torus knots. Next, we introduce some basic definitions and properties on gradings, filtrations, homological algebra and TQFT's. Our references are [Aud14], [Zha25], [Ati88], [Wei94].

1.1 Knots and links

1.1.1 Definitions

To prevent ambiguity and to simplify, all our knots and links are taken to be in $S^3 = \mathbb{R}^3 \cup \{\infty\}$, unless it is specified otherwise.

Definition 1.1. *An oriented knot K is a smooth embedding $K : S^1 \hookrightarrow S^3$. Without ambiguity, we take the knot K to be the image $K(S^1)$.*

Definition 1.2. *For $n \geq 1$ an integer, an oriented n-component link L is a disjoint union of n oriented knots $\sqcup_{i=1}^{n} S^1 \to S^3$. Hence a knot is a 1-component link.*

Definition 1.3. *Two links L_0 and L_1 are isotopic if there is a smooth map $\phi : \sqcup S^1 \times [0,1] \to S^3$ such that L_t is a link for all $t \in [0,1]$.*

One can check easily that the isotopy relation is an equivalence relation on the set of links.
From now on, we consider links up to isotopy unless it is specified otherwise. We call the isotopy classes a link or a knot without ambiguity.

Example 1.1. *Here is a first yet important example. A knot can be an unknot! We call the unknot the standard embedding of $S^1 \hookrightarrow S^3$ and denote it U.*

Here are some other well known knots and links

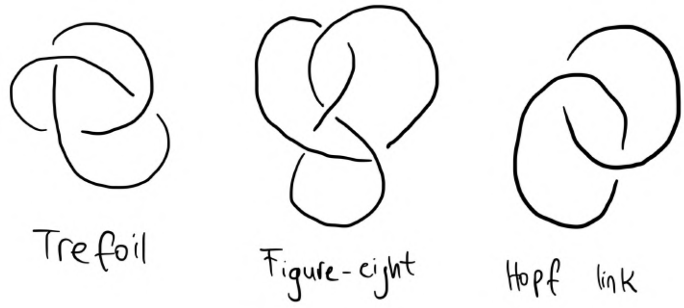

Figure 1.1: Trefoil knot, Figure-eight knot and Hopf link

Definition 1.4. *An orientation on a knot K is giving an orientation of the embedded circle. We can of course orient our links by orienting each knot component and consider them up to orientation preserving isotopy. An oriented link is thus an orientation preserving ambient isotopy class.*

1.1.2 Link diagrams

We represent a link by a link diagram which contain some crossing information.

Definition 1.5. *Let L be a link. A projection $\pi : S^3 \to \mathbb{R}^2$ such that $\pi \circ L$ is an immersion and such that $\pi \circ L$ is injective except at transverse double points is called generic. Indeed up to some isotopy any link admits a regular projection.*

Definition 1.6. *Given a link L and a generic projection, the double points of $\pi(L)$ are called crossings. If we identify the image $\pi(S^3)$ to the xy-plane, in a neighbourhood of a crossing we call the strand with bigger z-coordiante an overcrossing and the one with the smaller z-coordiante an undercrossing. A link diagram (or a planar diagram of L) consists of $\pi(L)$ with a deleted small neighbourhood of the strand corresponding to the under-strand.*

We can always associate a link diagram to a link. Conversely, we can always associate a link to a link diagram. Indeed in the given link diagram, we can glue an arc between the endings of the strand with deleted neigbourhood. In S^3 the arc will be under the crossing.

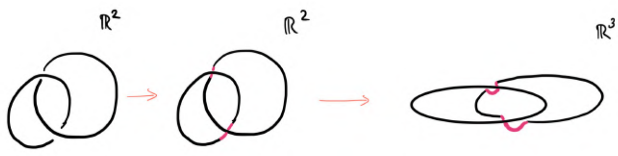

Figure 1.2

Definition 1.7. *An oriented link diagram is a link diagram with orientations on its knot components.*

Definition 1.8. *We say that two (oriented) link diagrams are equivalent if there is some (orientation-preserving) homeomorphism of* $\mathbb{R}^2$ *between them.*

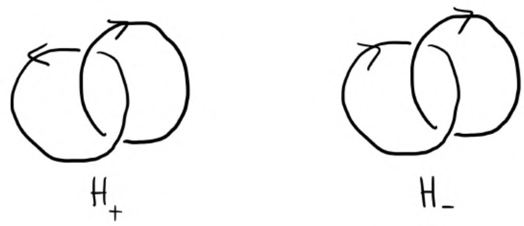

Figure 1.3: Two orientations for the Hopf link

As we said, link diagrams contain some crossing information. Moreover an oriented link diagram can tell us if a crossing is ”positive” or ”negative”:

Definition 1.9. *Let c be a crossing on an oriented link diagram. We say that c is positive or negative depending on one of the situations below:*

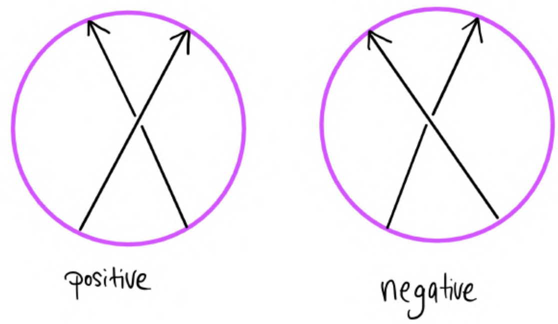

Figure 1.4: Positive and negative crossings

Definition 1.10. *A positive (respectively negative) link diagram is an oriented link diagram where all the crossings are positive (respectively negative).*

1.1.3 Reidemeister moves

We remark that a link can have more than one link diagram, so there is not only one way to draw a link. Indeed, two diagrams representing the same link will be related by some local moves called Reidemeister moves. That is to say that they will differ on some local sections by possible types of Reidemeister moves defined below.

Definition 1.11. *Reidemeister moves R1, R2 and R3 on a link diagram are local operations on link diagrams as showed below:*

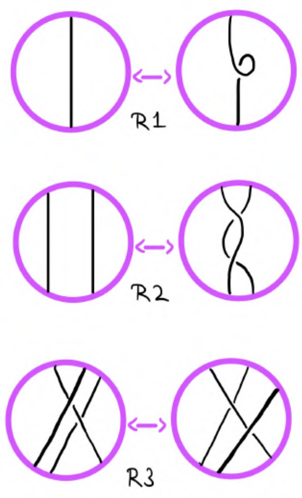

Figure 1.5: The three Reidemeister moves

Theorem 1.1 (Reidemeister)**.** *Two link diagrams represent the same link if and only if they are related by Reidemeister moves.*

For example below are two diagrams for the unknot and for the trefoil:

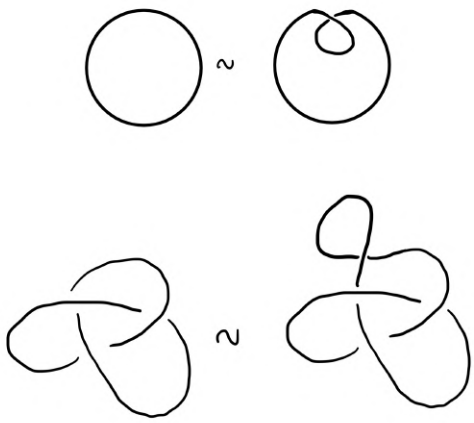

Figure 1.6: Different diagrams for the unknot and the trefoil

We have some operations on oriented knots as taking its mirror image or reversing its orientation.

Definition 1.12. *Let K be an oriented knot. The mirror image of K denoted $\bar{K}$ is the image of $r \circ K$ where r is an orientation reversing homeomorphism of S^3.*

Remark 1.1. *Any two orientation-reversing homeomorphisms are isotopic, so the mirror image of a knot is well defined.*

Remark 1.2. *For a knot K with a knot diagram D, we obtain a knot diagram $\bar{D}$ of $\bar{K}$ where we interchange the overcrossing with the undercrossings.*

Example 1.2. *Here is the mirror image of the right handed trefoil* 3_1

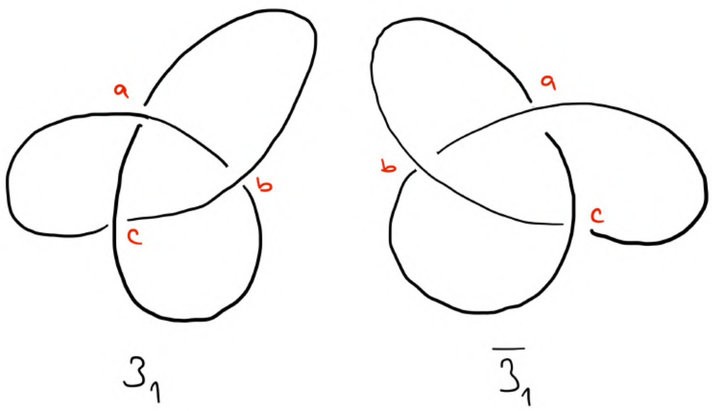

Figure 1.7: Mirror image of the Trefoil knot

Observe that a positive crossing becomes a negative crossing and vice versa after taking the mirror image of a knot.

If K is an oriented knot, denote by K^r the same knot with the reversed orientation. Observe that the signs of crossings do not change.

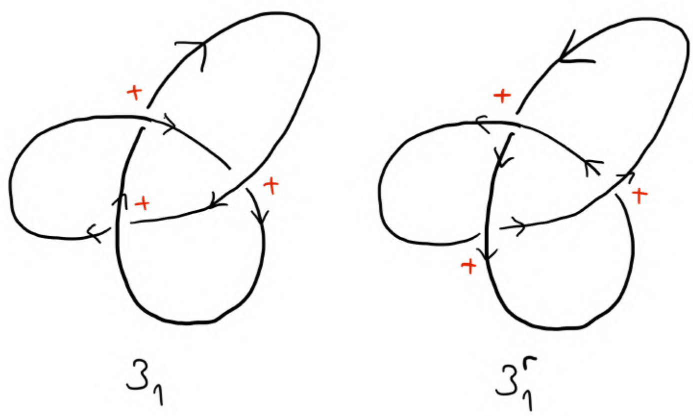

Figure 1.8: Invariance of signs of crossing under orientation reversion

1.1.4 Resolutions of crossings

As we said, a link diagram has crossings which correspond to the double points. Resolving those crossings encodes information about the link; the diagram resolution is often used in the process of getting link invariants and we will see some examples of how.

Definition 1.13. *We say that a crossing c is resolved (or smoothened) if on a neighbourhood of the crossing we change the link diagram in one of the following ways, called the 0-resolution or 1-resolution:*

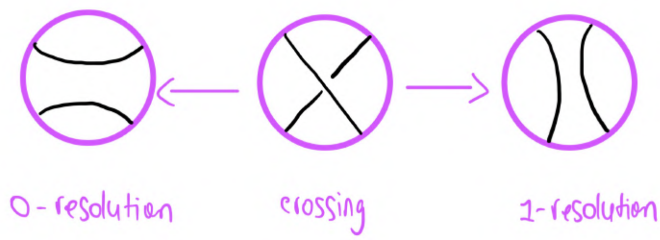

Figure 1.9: 0-resolution and 1-resolution

Definition 1.14. *Let D be a link diagram. A resolution of D is a choice of resolution for each crossing of D.*

For an oriented link diagram L, there is a way of obtaining a canonical resolution.

Definition 1.15. *Let L be an oriented link diagram. The oriented resolution of L consists of taking the orientation-friendly resolution at each crossing. In other terms, it is to take the resolutions as below so that the we obtain a collection of oriented and possibly nested circles.*

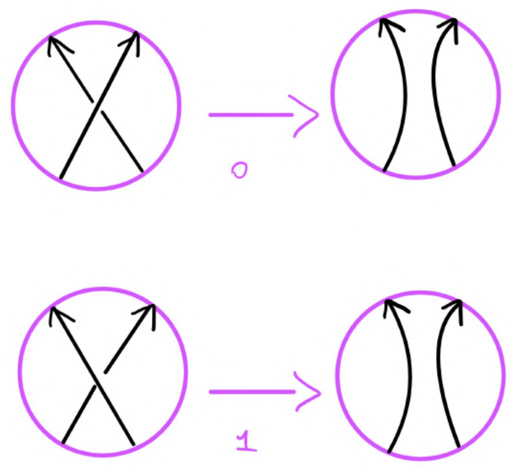

Figure 1.10: Oriented resolution

Example 1.3. *Let's order the three crossings of the trefoil as c_1, c_2, c_3. Then the $(0, 0, 1)$ resolution corresponds to the picture below*

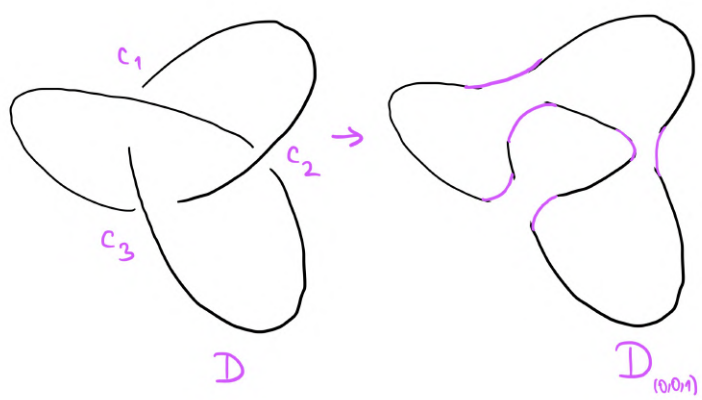

Figure 1.11: $(0, 0, 1)$ resolution of the trefoil

Example 1.4. *Now we orient the trefoil as below and we get the the oriented resolution below.*

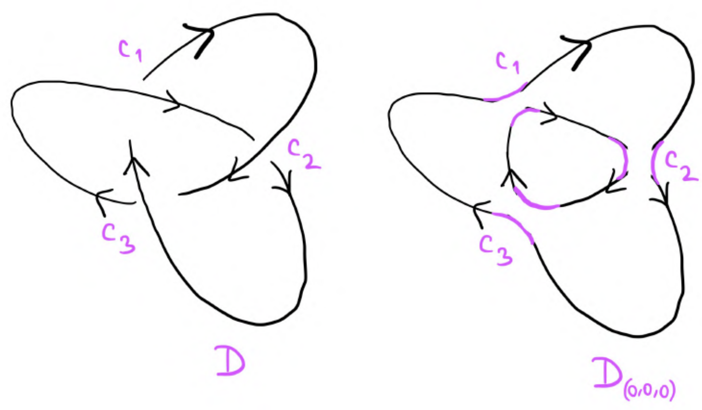

Figure 1.12: Oriented resolution of the trefoil

Hence a resolutions of a link diagram gives us a collection of possibly nested circles and the oriented resolution gives us oriented circles.

1.1.5 Knot and link invariants

Knot invariants are quantities or algebraic objects associated with knots that remain unchanged under ambient isotopy. They serve as fundamental tools for distinguishing knots and studying their topological properties. Examples include numerical invariants, polynomial invariants, and homological invariants, each capturing different aspects of knot structure.

Here are some classical examples of different types of link (or knot) invariants:

Numerical invariants:

- Linking number
- Crossing number
- Signature
- Arf invariant

Algebraic invariants:

- Alexander polynomial
- Jones polynomial
- HOMFLY-PT polynomial
- Seifert matrix
- Seifert form
- The knot group
- Knot Floer homology
- Khovanov homology

Topological invariants (up to homeomorphism)

- The complement of a link in S^3

The Rasmussen invariant s that we will define later is a numerical knot invariant and its definition relies on the Khovanov homology, which is a homological link diagram invariant.
Let us give some definitions, properties and examples of some of these invariants which we will mention throughout the paper.

Crossing number, Unknotting number, Linking matrix

A natural property to consider in a given link diagram is the number of crossings. It turns out that we can define a link invariant out of it.

Definition 1.16. *Let L be a link. The crossing number $c(L)$ is the minimal crossing number among all link diagrams of L.*

It is also natural to ask how one can transform a knot into the unknot, by applying some unknotting operations and see if we can also extend an invariant out of it.
At one of the crossing points of a diagram, exchange locally the over and under crossing segments. Since this type of alternation is not an elementary knot move, in general we obtain a diagram of some other knot.

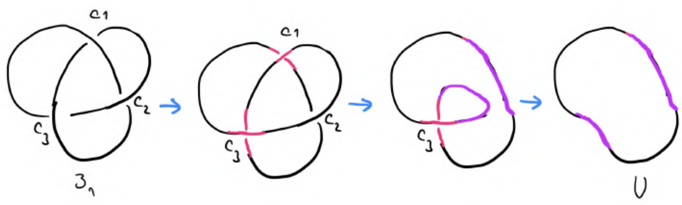

Figure 1.13: Unknotting moves on the trefoil

Proposition 1.1. *We can change a diagram D of an arbitrary knot to the diagram of the trivial knot by exchanging the over/under crossing segments at several crossing points of D, and using several Reidemeister moves. The above operation is called the unknotting operation.*

Definition 1.17. *We define the unknotting number of D as the minimum number of unknotting operations that is required to change D into the diagram of trivial knot.*

Theorem 1.2. *If K is a knot, then $u(K) = \min_D u(D)$ is a knot invariant.*

In fact, the two knot invariants that we discussed previously are independent of the assigned orientation of the knot. Happily there are invariants which distinguish between two oriented knots. For example the linking number is an invariant of oriented links defined as follows.
Let L be an oriented link and D an oriented diagram. We can talk about a crossing being positive or negative. We assign to each crossing point of the diagram $+1$ or -1, according to the sign of the crossing.

Definition 1.18. *If K_1 and K_2 are two disjoint knots whose projections intersect at points $c_1, \dots, c_m$, define their linking number as*

$$lk(K_1, K_2) := \frac{1}{2}(sign(c_1) + \cdots sign(c_m))$$

Theorem 1.3. *$lk(K_1, K_2)$ is an invariant of $L = K_1 \cup K_2$. Moreover, $lk(K_1, K_2^r) = -lk(K_1, K_2)$.*

Definition 1.19. *Now, if L has n components $K_1, \cdots, K_n$, define the total linking number of L by*

$$lk(L) := \sum_{i<j} lk(K_i, K_j)$$

1.1.6 The connected sum and concordance group of knots

We may define a sum operation on the set containing all the knots. If via such an operation we could get a group, then we could adopt group theoretical method to study knots.
Consider a 3-ball in S^3 and a sphere S bounding it. In B^3, take a single curved line α whose endpoints A, B are on S. If it intersects S only at A, B then it is called a $(1,1)$ tangle. Note that a $(1,1)$ tangle can have disjoint, single, closed curves.
Apply elementary knot moves to the segments of the $(1,1)$ tangle α that lie in the interior of B^3. Doing so, we change α, having fixed A, B as shown below:

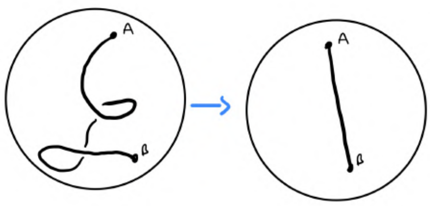

Figure 1.14: Applying elementary moves to the $(1,1)$ tangle α

Suppose K is a knot and that there exists a 2-dimensional sphere Σ intersecting K exactly at two points: A, B. We think of Σ replacing the role of S above. But since K lies in S^2, it divides Σ into 2 $(1,1)$ tangles. Indeed, think of S^3 as two 3 balls glued along their boundaries. Then a $(1,1)$ tangle α is formed from Σ and its interior, and another $(1,1)$ tangle β is formed from Σ and its exterior.
Now, connect A to B by a simple polygonal line s that lies on Σ. Joining s to α we get a knot K_1, joining s to β we get a knot K_2. Here we have shown that K can be decomposed into two knots. Indeed, any knot can be decomposed to what we call prime knots; a prime knot is a knot which does not admit such a decomposition without the

trivial tangle.
Take a point P on an oriented knot, and think of it as the center of a ball B^3 with very small radius and having the following two properties:

(1) K touches exactly 2 points on the surface of boundary sphere of B^3

(2) In the interior of B^3, the $(1,1)$ tangle α that is obtained from K is a trivial tangle.

Let K' be another knot in another S^3 and similarly take P' and a trivial tangle β'. We can assign orientation naturally to the tangles, induces by that of the knots.
Removing from S^3 the interior points of the 3-balls, we get two 3-balls whose boundaries are S^2 in S^3. Then by gluing these balls on their boundaries via an orientation-reversing homeomorphisms on one of the S^2s, we obtain a new S^3. In this process, the end point of α and the initial point of β' are joined.

Definition 1.20. *Adopting the same notations in the above discussion, we get a new knot (in a new S^3 !) called the connected sum $K_1 \# K_2$. This knot is oriented by construction, and in fact doesn't depend on the points P, P', hence totally determined by K_1, K_2.*

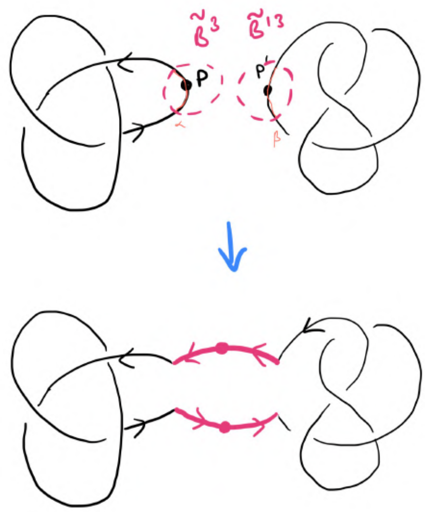

Figure 1.15: The connected sum of the trefoil knot and the figure-eight knot

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We can extend this definition to links, but it depends on what components the connected sum occurs.

Definition 1.21. *The connected sum of two links is obtained by taking a connected sum of two of their knot components. Of course this depends on the chosen components.*

Remark 1.3. *The connected sum operation does not depend on the choice of the balls but on the orientations of the knots, as in the connected sum operation of oriented manifolds.*

The connected sum operation seems suitable to get a group out of the set of knots. Indeed, it is associative, has the trivial knot as the unit and is even commutative:

Proposition 1.2.
- $K_1\#(K_2\#K_3) = (K_1\#K_2)\#K_3$
- $K_1\#K_2 = K_2\#K_1$
- $K\#U = K$

But it turns out that no non trivial knot admits an inverse. To ameliorate this situation, in the 1950's Milnor introduced a new equivalence relation between knots, today called concordance. With respect to this definition, the set of all knots does become a group under the connected sum operation defined above. The knots which are equivalent to the unknot are called *slice*.
To explain the idea of a slice knot, consider S^3 as ∂B^4.

Definition 1.22. *K is slice if it is the boundary of a disk D in B^4, not having any singular points. A point is not singular if we can choose a neighbourhood U, diffeomorphic to a 4-dimensional ball, and such that $(U, D) \cong (\mathbb{R}^4, \mathbb{R}^2)$ and that $\partial U \cap D$ is a trivial knot in ∂U, which is a 3-sphere.*

Definition 1.23. *A concordance between knots K_0 and K_1 is a cylinder $C = S^1 \times [0, 1]$ smoothly embedded in $S^3 \times [0, 1]$ in such a way that the ends $S^1 \times \{i\}$ are embedded in $S^3 \times [0, 1]$ as $K_i \times \{i\}$ and such that C has a tubular neighbourhood $C \times D^2$ intersecting $S^3 \times \{i\}$ in tubular neighbourhood of $K_i \times \{i\}$.*

We say that two knots are *concordant* if they are connected by a concordance. A knot is said to be *slice* if it is concordant to the unknot U.

Definition 1.24. *The concordance group denoted $Conc(S^3)$ is the quotient group of oriented knots under the concordance relationship where the group operation is the oriented connected sum operation, the identity element is the unknot U and the inverse of an oriented knot K is given by $(K)^{-1} = \bar{K}$. It is an abelian group. Indeed if S is a concordance from K_0 to K_1 then S^r is a concordance from K_1 to K_0.*

Definition 1.25. *The slice genus $g_*(K)$ of a knot K is the minimal genus among all surfaces that K bounds and which are smoothly properly embedded in B^4*

Example 1.5. *The square knot above is slice but the trefoil and the figure eight are not slice.*

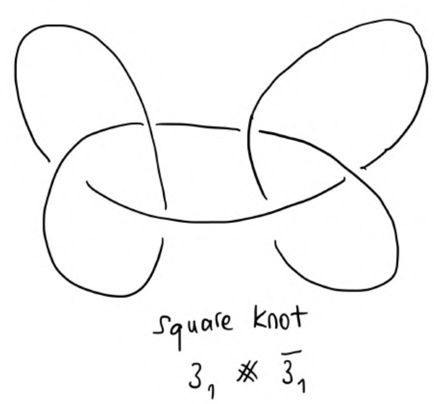


Figure 1.16: The square knot

1.1.7 Seifert matrix

The Alexander polynomial, discovered by J.W. Alexander in 1928 has been one of the most important cornerstones in knot theory.
The study of the Alexander polynomial and the determination of its precise context has been extensively carried out. It has been found out that the Alexander polynomial is closely related to the topological properties of the knot and had a great impact on the theory of knots. There are various methods to compute and to define the Alexander polynomial. In the mechanic of one of these methods, a concept called the Seifert matrix is brought up to light and for our purposes, this definition and method of computing the Alexander polynomial will suffice.

The following theorem from 1930's is due to Pontrjagin and Frankl.

Theorem 1.4. *Given an arbitrary oriented knot K, there exists in $\mathbb{R}^3$ an oriented connected surface S such that $\partial F = K$.*

We give Seifert's proof, which is a constructive proof called *Seifert's algorithm.* Given a knot K, such a surface obtained by the Seifert's algorithm is called a Seifert surface.

Proof. The idea is to take a diagram D and to decompose it into several closed curves and fill them in and to re-obtain the knot as boundary using twisted bands. The orientability condition will be satisfied by some colourability of the surface.

Step 1: Let D be an oriented link diagram. We consider the oriented resolution of D. As we said before, the oriented resolution gives us a collection of oriented and possibly nested circles in the plane, called *Seifert circles.* Moreover for nested circles in S^3, one can consider that the bounded disks are all disjoint as follows: for a circle C define its *nested height* $h(C)$ by the number of circles one must cross to get to the point at infinity in the diagram and fill each circle C by a disk parallel to the xy-plane at the height $h(C)$.

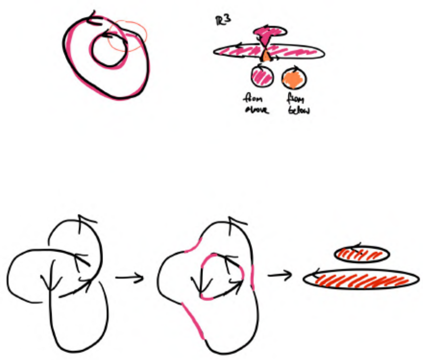


Figure 1.17: Embedding the nested circles in R^3

Step 2: Now we will choose a bicoloring of these disks as follows:
Colour a disk in red or green such that if its boundary runs counterclockwise seen by the red side of the disk and colour in green the other side. For example if a disk is oriented counterclockwise then the upper side of the disk will be red.

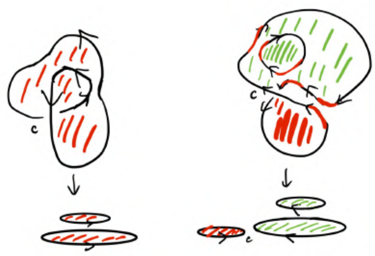


Figure 1.18: Embedding the nested circles in R^3

Step 3: Now for a crossing c, resolve every other crossing but c. There are two possible diagrams one can obtain and resolving the crossing c one gets a pair of nested circles with the same colour facing up or unnested circles with different colours as in the figure below.

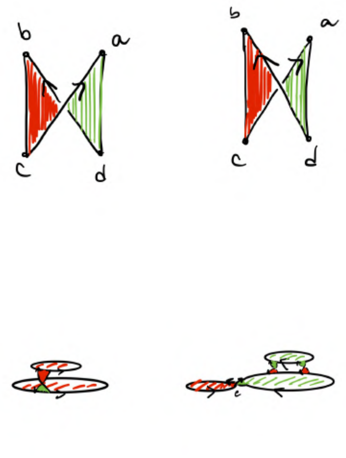


Figure 1.19: Joining bands to disks

Then we join the disks by half twisted bicoloured strips such that the colours of the feet of the handles are compatible with the colours of the disks they are attached. Indeed, this is adding a half twisted 1-handle for each crossing as shown in the figure below. If D is a knot than the surface will be connected and if D is a link we join the connected components by tubes. We have then a connected, compact and oriented surface bounded by the link. □

We recall the classification of closed oriented, connected surfaces:

Theorem 1.5. *Let S be a closed, oriented, connected surface. Then S is homeomorphic to S^2 or to a connected sum of g tori where $g \in \mathbb{Z}_{\geq 0}$ is the genus of the surface.*

This fact together with the existence of a Seifert surface for every link allows us to define the genus of a link.

Definition 1.26. *Let L be a link. The 3-genus $g(L)$ of L is defined as the minimal genus of all of its Seifert surfaces. That is*

$$g(K) = min\{g(S) \mid S \textit{ is a Seifert surface for } K\}$$

Clearly, a disk is a Seifert surface for the unknot U and hence $g(U) = 0$. The converse is true by the following proposition:

Proposition 1.3. *If K bounds an embedded disk, then K is the unknot.*

For an arbitrary knot, there exists an algorithm to actually compute its genus, but it is difficult to implement. In truth to compute the genus of an arbitrary knot is a difficult undertaking, however for certain types of knots, the calculation of the genus is relatively straightforward as we shall see.
Note that the Seifert surface associated to the diagram D, we have :

- points: each band has 4 vertices
- edges: polygonal curves that constitue the edges of the bands and the boundaries of the disks
- faces: disks and bands

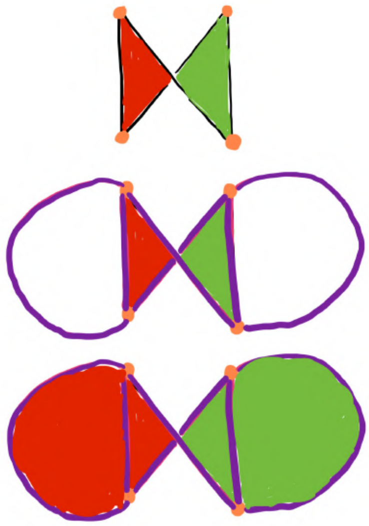


Figure 1.20: Vertices, edges and faces in the Seifert surface

Indeed, the genus could be computed a regular diagram D in the following way: Let d be the number of disks and b the number of bands in the Seifert surface. By the definition of the Euler characteristic, we get

$$\begin{aligned}\chi(S) &= 4b - 6b + d + b \\ &= d - b\end{aligned}$$

If L is a link with n components and S its seifert surface then

$$\chi(S) = 2 - 2g - n$$

Hence for a knot we get:

$$2g = 1 - d + b$$

Proposition 1.4. *Let K be a knot and S a Seifert surface. If S of genus g is constructed using b bands and d disks,*

$$2g = 1 - d + b$$

Another way of computing the genus of a Seifert surface is by considering what we call *the Seifert graph* associated to a diagram of a knot.

Let S be a Seifert surface of a knot K obtained from disks and bands as described above. Now shrunk each disk to a point, hence the widths are shrunk into segments. This gives us a planar graph, called the Seifert graph associated to D which we denote $\Gamma(D)$.

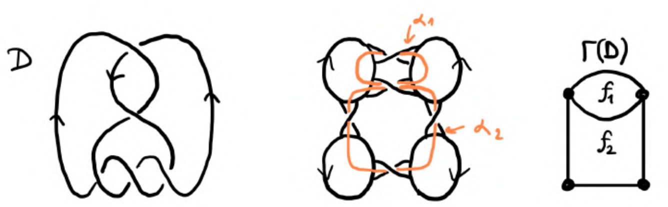


Figure 1.21: Seifert graph of a knot diagram D

$\Gamma(D)$ divides S^2 into several domains. In this partition we have d points and b edges and f faces. Then

$$2 = \chi(S^2) = d - b + f$$

Hence

$$f - 1 = 1 - d + b$$

So $1 - d + b$ is equal to the number of faces, excluding the region containing the point at infinity.

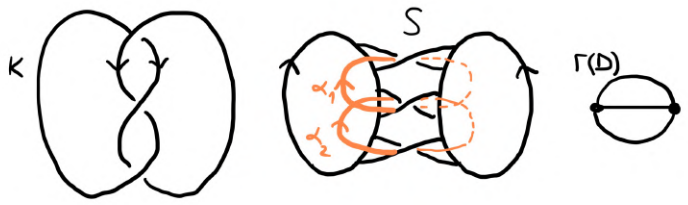


Figure 1.22

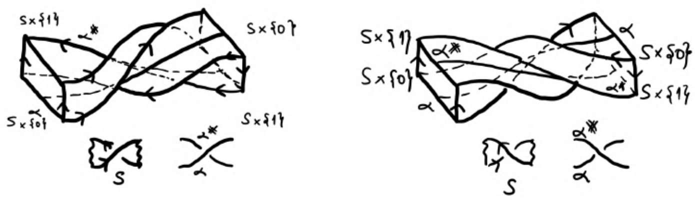


Figure 1.23: Liftings of the curves

Let D be a diagram of a knot K , $\Gamma(D)$ a Seifert graph and S the associated Seifert surface. We want to create $2g = f - 1$ closed curves on S, where f is the number of faces in the partition of S^2. Indeed, the boundary of each face except that containing the point at infinity, is a closed curve of $\Gamma(D)$. These curves hide some information about the knot K which is the boundary of the surface S. Individually, for sure they are of little interest. But as a collection, they provide us with a knot invariant.
For example as showed above, the figure eight knot has a Seifert surface of genus 1 amd $\Gamma(D)$ gives two closed curves $\alpha_1, \alpha_2^{\#}$.
It is possible that $\alpha_1 \cap \alpha_2 \neq \emptyset$, therefore the union of these curves is not always a link.

However, we can lift α_2 above the surface to a link parallel to α_2, we denote the lifted one $\alpha_2^{\#}$. Then, $\{\alpha_1, \alpha_2\}$ is a link. Here is the mathematical construction of $\alpha_2^{\#}$:
We thicken the surface S slightly: $S \times [0,1]$. Now we consider that the original surface S lies on the bottom of the thickened surface i.e. $S = S \times \{0\}$. Then we denote still α_1 and α_2 the curves $\alpha_i \times \{0\}$. Similarly we denote the lifted curves $\alpha_i \times \{1\}$ as $\alpha_i^{\#}$. Any orientations on α_i induce orientations on $\alpha_i^{\#}$. Hence we can compute the following matrix for the figure eight knot's given diagram.

$$\begin{pmatrix} lk(\alpha_1, \alpha_1^{\#}) & lk(\alpha_1, \alpha_2^{\#}) \\ lk(\alpha_2, \alpha_1^{\#}) & lk(\alpha_2, \alpha_2^{\#}) \end{pmatrix}$$

Definition 1.27. *For any knot K and Seifert surface S with genus g obtained by a diagram D, we define the Seifert matrix of the diagram D as the square and integer matix*

$$M := (lk(\alpha_i, \alpha_j^{\#})_{1\leq i<j\leq 2g}$$

By convention, if K is the trivial knot so if $g(S) = 0$ we take the Seifert matrix to be the empty matrix.

S-equivalence of Seifert matrices

The construction of the Seifert matrix above depends on the diagram D. If we want to use Seifert matrices, we should examine the effects of the Reidemeister moves on the Seifert matrices.

Theorem 1.6. *Two Seifert matrices obtained from equivalent knots K_1, K_2 can be changed from one to the other by applying a finite number of times the following operations, and they are said to be S-equivalent*

$$\Lambda_1 : M_1 \mapsto PM_1P^T$$
$$\Lambda_2 : M_1 \mapsto M_2$$

where P is an invertible integer matrice of $\det = \pm 1$ *and where M_2 is either one of the matrices below:*

$$\begin{pmatrix} & & & * & 0 \\ & M_1 & & \vdots & \vdots \\ & & & * & 0 \\ 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & \cdots & 0 & 0 \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} & & & 0 & 0 \\ & M_1 & & \vdots & \vdots \\ & & & 0 & 0 \\ * & \cdots & * & 0 & 0 \\ 0 & \cdots & 0 & 1 & 0 \end{pmatrix}$$

Figure 1.24

Remark 1.4. *Here, Λ_1 corresponds to the change of order/orientation in the diagram and Λ_2 corresponds to the change of the genus of the Seifert surface due to a Reidemeister move, hence the matrix will be bigger or smaller.*

Proposition 1.5. *Two Seifert matrices which represent the same knot are S-equivalent.*

Properties of Seifert matrices

Here are some basic properties of Seifert matrices:

Proposition 1.6. *Suppose K is an oriented knot. If K^r is the K with reversed orientation,*

$$M_K \sim M_K^T$$

Proof. Indeed, if D is a diagram for K then D^r where all the orientations are reversed is a diagram for K^r. Then, the Seifert surfaces have opposite orientations and the under/over crossings are exchanged, hence the relations for $\alpha_i \alpha_j^{\#}$ are reversed. $\square$

Proposition 1.7. *Let K be a knot and $\bar{K}$ be the mirror image of K. Then*

$$M_{\bar{K}} \sim -M_K^T$$

1.1.8 Invariants from the Seifert matrix

Hence, to find a knot invariant from the Seifert matrices, we need to look for something which will not change under operations Λ_1 and $\Lambda_2^{\pm 1}$.

2 examples of those invariants that we will consider are the Alexander polynomial and the signature.

The Alexander polynomial

It is natural to ask, having such a tool as Seifert matrix, what properties of it via Λ_i yield a knot invariant.
First we symmetrize the matrix M, forming $M + M^T$.

Proposition 1.8. *If M is a Seifert matrix of a knot K, then*

$$|det(M + M^T)|$$

is an invariant of K, and is called the determinant of the knot K.

A fact is that the determinant is also an invariant of unoriented knots, i.e. it does not change if we change the orientation of K.
The determinant is an old invariant, since $det(U) = 1$ it has been used to show if a knot is trivial. For example the trefoil knot has $det(3_1) = 3$, but there are other knots not equivalent to the trefoil knot yet have $det = 3$ so the determinant does not detect the trefoil.

Proposition 1.9. *Let M be a Seifert matrix of a knot K*

$$\det(M - M^T) = 1$$

At this stage, we jump directly from the above determinant to consider the polynomial:

$$\Delta_M(t) := det(M - tM^T)$$

Now we have a polynomial with indeterminate t. The next natural thing to do is to understand how this polynomial changes under Λ_i and their inverses.

Theorem 1.7. *Suppose M_1, M_2 are Seifert matrices for a knot K. Then if r and s are their orders, we have*

$$t^{-\frac{r}{2}} \det(M_1 - tM_1^T) = t^{-\frac{s}{2}} \det(M_2 - tM_2^T)$$

In other terms, $t^{-\frac{n}{2}} det(M - tM^T)$ is a knot invariant if M is a Seifert matrix of order n for K. This drives us to the following definition

Definition 1.28. *We define the Alexander polynomial of the knot K as*

$$\Delta_K(t) = t^{\frac{-n}{2}} det(M - tM^T)$$

Remark 1.5. *The whole discussion above, starting from the Seifert matrix, is valid for links. Hence this definition of Alexander polynomial is also valid for links.*

By the previous discusssion, for a knot we have that the order k of the matrix is

$$k = 2g(S)$$

If we multiply $\Delta_K(t)$ by a suitable factor, we obtain a polynomial with only positive exponents.

Here is an important property of the Alexander polynomial.

Theorem 1.8. *Suppose K is a knot. Then $\Delta_K(t)$ is a symmetric Laurent polynomial : $\Delta_K(t) = \Delta_K(-t)$.*

$$\Delta_K(t) = a_{-n}t^{-n} + a_{-n+1}t^{-n+1} + \cdots + a_{n-1}t^{n-1} + a_nt^n$$

with

$$a_{-i} = a_i$$

Proposition 1.10. *Let K be a knot, then*

$$|\Delta_K(-1)| = det(K)$$

Now let us see some basic but important properties of the Alexander polynomial.

Proposition 1.11. *If K is a knot, then $\Delta(1) = 1$.*

We can characterize the Alexander polynomial of a knot by the following theorem.

Theorem 1.9. *Let f be a Laurent polynomial such that*

- $f(1) = 1$
- $f(t) = f(t^{-1})$

Then there exists a knot K such that $\Delta_K(t) = f(t)$

Theorem 1.10. *Let K be a knot.*

(1) Let K^r be K with reversed orientation. Then $\Delta_K(t) = \Delta_{-K}(t)$

(2) $\Delta_{\bar{K}}(t) = \Delta_K(t)$

Remark 1.6. *Also, remark that the Alexander polynomial does not distinguish the trivial knot. There exists non-trivial knots with Alexander polynomial being 1, for example the Kinoshita-Terasaka knot below*

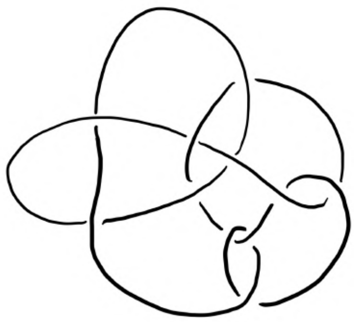


Figure 1.25: Kinoshita-Terasaka knot

Theorem 1.11. *Let K_1, K_2 be two knots. Then*

$$\Delta_{K_1 \# K_2} = \Delta_{K_1} \Delta_{K_2}$$

Now, we come back to the difficult problem of determining the genus of a knot. As we said, in some cases it is possible to estimate or even to determine the genus of a knot using the Alexander polynomial.
Suppose that for a knot K of genus g, S is a Seifert surface with genus g. Then, the order of the Seifert matrix for that Seifert surface will be $2g$. Therefore, the maximum degree of t in $\Delta_K(t)$ is $\leq g$.

Theorem 1.12. *Let K be a knot with genus g. Then the maximum degree of t in $\Delta_K(t)$ cannot exceed g.*

In particular, if $det(M) \neq 0$, since $det(M - M^T)$ is a Laurent polynomial of exactly degree $2g$ we get:

Proposition 1.12. *A necessary and sufficient condition for the maximum degree of t in $\Delta_K(t)$ to be g is $det(M) \neq 0$.*

Then, for example, the genus of an alternating knot is equal to the maximum degree of Δ_K.

The signature of a knot

Another important invariant that Seifert matrices give us is the signature of a knot. Indeed, $|\det(M + M^T)|$ is not the only invariant that one can form from the Seifert matrix.

Note that if $M \in M_k(\mathbb{Z})$ is a matrix than $M + M^T \in M_k(\mathbb{Z})$ is a symmetric matrix. Recall some basic results about real symmetric matrices, conducing to notion of signature of a symmetric matrix.
By linear algebra, we know that every symmetric matrix with real entries is diagonalizable.

Theorem 1.13 (Sylvester's theorem)**.** *Two diagonal matrices in $M_k(\mathbb{R})$ are congruent if and only if they have the same number of positive, negative and zero entries.*

So each symmetric matrix will be congruent to some $diag(1, 1, ..., 1, -1, -1, ..., -1, 0, 0, ...0)$ with $\#|1| = p, \#|-1| = n$

Definition 1.29. *The number $\sigma = p - n$ is called the signature of the matrix*

In particular, two invertible symmetric matrices are congruent if and only if they have the same signature.

Theorem 1.14. *If N_1 and N_2 are two integer square matrices that are S-equivalent, then*

$$\sigma(N_1 + N_1^T) = \sigma(N_2 + N_2^T)$$

Definition 1.30. *Let K be a knot and M be a Seifert matrix. Then the signature of K is*

$$\sigma(K) := \sigma(M + M^T)$$

Computing the signature is a very effective way of rooting out the local properties of a knot: it has a profound influence in the solution of local problems.
We look at the problem of amphichirality for example. we say that a knot K is amphichiral if it is equivalent to its mirror image. To distinguish a knot from its mirror image is a very difficult problem. To show that the right-handed trefoil 3_1 is not equivalent to its mirror image required an elaborative proof at the time.

Moreover, the signature does not detect the orientation reversing and behaves well with the connected sum.

Proposition 1.13.

- *If $r : S^3 \to S^3$ is an orientation reversing homeomorphism, $\sigma(K^r) = \sigma(K)$*
- $\sigma(\bar{K}) = -\sigma(K)$
- *$\sigma(K)$ is an even number.*
- $\sigma(K_1 \# K_2) = \sigma(K_1) + \sigma(K_2)$

Theorem 1.15. *Let K be a knot and D a diagram. Then $\sigma(K)$ can be determined by means of the following three axioms:*

(1) $\sigma(U) = 0$

(2) D_+, D_- differ by a crossing change, then

$$\sigma(D_+) = \sigma(D_-) \text{ or } \sigma(D_+) = \sigma(D_-) - 2$$

(3) If Δ_K is the Alexander polynomial of K, $sign(\Delta(-1)) = (-1)^{\frac{\sigma(K)}{2}}$.

Example 1.6. *Let us illustrate this theorem on the figure eight knot.*

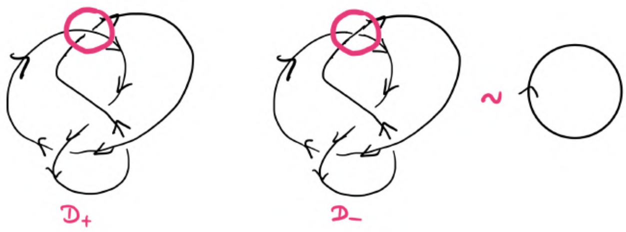


Figure 1.26: Skein diagrams for the figure-eight knot

Hence

$$\sigma(D_+) = \sigma(D_-) = 0 \text{ or } \sigma(D_+) = -2$$

Since $\Delta_K(t) = -t^{-1} + 3 - t$ and $\Delta_K(-1) = 5$ so $sign(\Delta_K(-1)) = 1$. Then we must have $\sigma(K) = 0$.

This was expected since it is known that

$$4_1 \sim 4_1^*$$

If we perform a single unknotting operation on a knot K, we change D_+ to D_- or D_- to D_+. Then the signature remains the same or changes by ± 2. If $u(K)$ is the unknotting number of the knot K, by performing $u(K)$ unknottings we will get the trivial knot, hence we get the following theorem. Note that this theorem is not sufficient to determine the unknotting number and the determination of the unknotting number for an arbitrary knot remains a difficult problem.

Theorem 1.16. *If $u(K)$ is the unknotting number of a knot K,*

$$\sigma(K) \leq 2u(K)$$

The signature gives an osbtruction to being topologically slice. The following theorem is due to Kauffman and Taylor. [KT76].

Theorem 1.17. *Let K be a knot. Then $\sigma(K) \leq 2g_{top}(K)$ where $g_{top}(K)$ is the topological slice genus of K.*

As the following theorem suggest, the signature of a knot induces a homomorphism $Conc(S^3) \to \mathbb{Z}$

Theorem 1.18. *Concordant knots have the same signature. Indeed one has a surjective homomorphism of abelian groups*

$$Conc(S^3) \to 2\mathbb{Z}$$

Proof. Let K_0 and K_1 be concordant knots. This means that $K_0^r \# K_1$ is slice. Hence

$$\sigma(K_0^r \# K_1) = 0$$
$$\sigma(K_1) - \sigma(K_0^r) = 0$$

Now recall that the right-handed trefoil has Seifert matrix $\begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix}$. Then $\sigma(3_1) = -2$. Then the left handed trefoil has signature $+2$ and by the additivity of the signature we can realize any even integer as the signature of a knot. Clearly the signature is even since the order of a Seifert matrix of a knot K is $2g$ where g is the genus of the corresponding Seifert surface. $\square$

Corollary 1.1. *If $\sigma(K) \neq 0$ then K has an infinite order in $Conc(S^3)$.*

Example 1.7. *The figure 8 knot 4_1 represents an element of order 2 in $Conc(S^3)$. Indeed 4_1 is not slice but we have that $[4_1] = [\bar{4_1}^r]$. Hence:*

$$\begin{aligned} 2[4_1] &= [4_1] + [\bar{4_1}^r] \\ &= [4_1 \# \bar{4_1}^r] \\ &= 0 \end{aligned}$$

The Jones Polynomial via the Kauffman bracket

The Jones polynomial is a Laurent polynomial invariant of oriented knots and links discovered by Jones in 1984. It is defined recursively by a skein relation and can distinguish many knots that share the same classical invariants.

Definition 1.31. *The Kauffman bracket of a link diagram D is given by the following recursive formula*

- $\langle \times \rangle = \langle)(\rangle - q \langle \asymp \rangle$
- $\langle kcircles \rangle = (q + q^{-1})^k$

The Kauffman bracket is not a link invariant, but we define the unnormalized Jones polynomial which is a link invariant

Definition 1.32.

$$\hat{J}(D) = (-1)^{n_-} q^{n_+ - 2n_-} \langle D \rangle$$

Definition 1.33. *The Jones polynomial of a link L is*

$$J(L) = \frac{\hat{J}(L)}{(q + q^{-1})}$$

To find the Jones polynomial of a link diagram D, we replace in the cube of resolutions the vertices by the corresponding polynomials.
Let $r \in \{0,1\}^n$ be a smoothing of D and denote D_r the resolved diagram and Γ_r the corresponding Laurent polynomial $\in \mathbb{Z}[q^{\pm 1}]$. Denote

- $|r|$ the number of 1's in r
- $|D_r|$ the number of circles in D_r

then by the recursive definition we have

$$\hat{J}(D) = \sum_{r \in \{0,1\}^n} (-1)^{|r| - n_-(D)} \; q^{|r| + n_+(D) - 2n_-(D)} (q + q^{-1})^{|D_r|}$$

Example 1.8. *Here is an example of computation for the Hopf link*

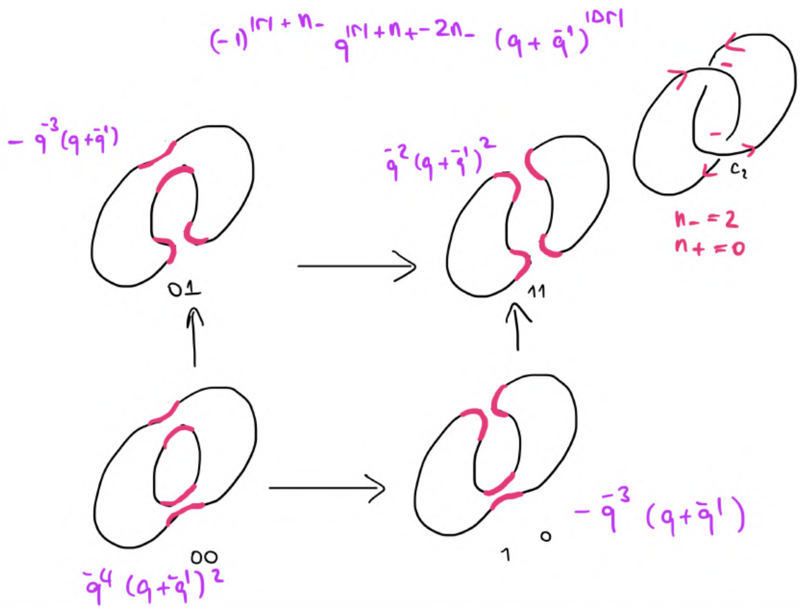


Figure 1.27

Then we get the unnormalized Jones polynomial of the Hopf link

$$\hat{J}(H_-) = q^{-6} + q^{-4} + q^{-2} + 1$$

1.1.9 Torus Knots

In this section, we want to investigate an important class of knots. They are also the object of the special case of the Milnor conjecture that Rasmussen proved in his paper, which we will study in Chapter 2.
Attention, even though torus knots are already classified as we shall see here, we should underline that the methods which allow the classification are not sufficient to totally determine all the properties of the torus knots. For example as we shall see, the invariants such as slice genus and the unknotting number were objects of relatively recent studies. Hence, the solution of local problems does not follow automatically from that of global problems.
In all the following, suppose that q, r are relatively prime integers unless it is specified otherwise.

Definitions and properties

Definition 1.34. *The trivial torus T is obtained by rotating around the y-axis the unit circle centered at* $(2, 0)$

$$m : (x-2)^2 + y^2 = 1$$

on the xy-plane, in $\mathbb{R}^3$.

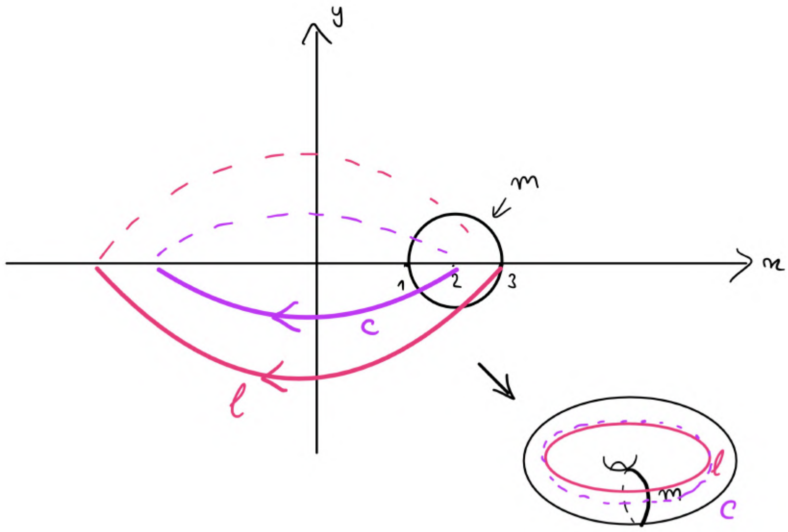


Figure 1.28: Obtainingthe trivial (solid) torus

Rotating the point $(3,0)$ around the $y-$axis, we obtain another circle l on the surface. We say that the circle m is the meridian and the circle l is the longitude of the torus, up to the elementary knot moves. Moreover, the circle C obtained by rotating the center $(2,0)$ is the center of the torus T. Notice that l and C are parallel where C and m form a link with $lk(m,C)=1$ under some orientations. We orient l same as C.

An alternative way to construct the trivial torus is to take a cylinder in $\mathbb{R}^3$ with the unit circles C_1 as its base and C_2 as its top and gluing them so that the central axis C becomes the trivial knot. A knot that lies on the trivial torus is said to be a torus knot.

Definition 1.35. *A knot K is said to be a torus knot if it is equivalent to a knot that can be drawn without any points of intersection on the trivial torus.*

Following the alternative definition described above, we can construct a torus knot as follows:
Take $2r$ points on the boundary circles of the cylinder:

$$A_0,\ldots,A_{r-1}\in C_1$$
$$B_0,\ldots,B_{r-1}\in\ C_2$$

with the given coordinates:

$$A_i=\Big(\cos\frac{2\pi i}{r},\sin\frac{2\pi i}{r},0\Big)$$
$$B_i=\Big(\cos\frac{2\pi i}{r},\sin\frac{2\pi i}{r},1\Big)$$

Now connect the point A_k and B_k on the cylinder by segments α_k and keeping the base, give the cylinder a twist by rotating the top about the z-axis by an angle $\frac{2\pi q}{r}$ with some q a non-zero integer.

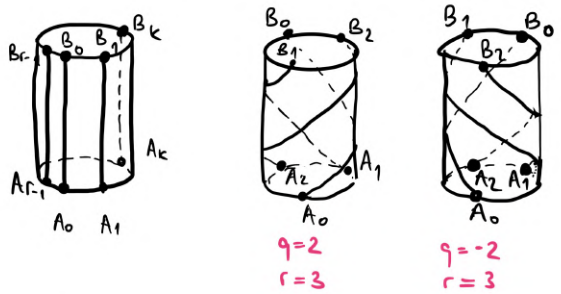


Figure 1.29: Obtaining a torus knot $T_{q,r}$

Finally glue the circles by identifying the points $(x, y, 0)$ and $(x, y, 1)$. This creates a single trivial torus T with the r segments α_i that have been transformed into a knot on its surface. We can assign an orientation to the obtained knot by assigning an orientation to each segment α_i. If we give each α_i the orientation going from A_i to B_i, we denote the resulting oriented knot by $T_{q,r}$.
If we reverse the orientation of each α_i then we denote the oriented knot by $T_{-q,-r}$. Indeed, the knot $T_{q,r}$ has the following knot diagram

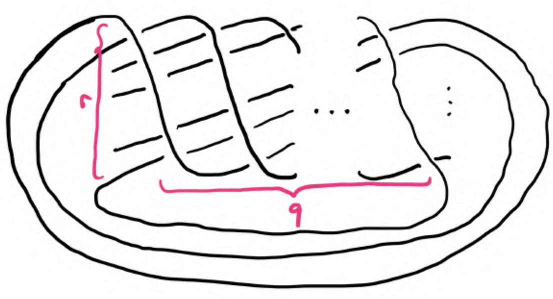


Figure 1.30: A diagram for the torus knot $T_{q,r}$

As we see, the knot $T_{q,r}$ wraps the surface following the meridian q times and wraps following the longitude r times.
Finally, notice that the circle m can be then viewed as the torus knot $T_{1,0}$ and a circle of a boundary component on the torus can be viewed as $T_{0,0}$.

Classification of torus knots

Torus knots were first systematically studied in the 19th century in connection with the topology of curves on the torus and links arising from algebraic geometry. Their classification as knots T(p,q) with coprime integers p,q reflects the fact that they wrap p and q times around the two standard cycles of the torus. Early work in knot theory, particularly in the late 1800s, already recognized that these knots form a distinguished family among prime knots. By the mid-20th century, their role became central through connections with singularity theory and algebraic curves. This led to their modern classification as the simplest infinite family of nontrivial knots admitting a complete arithmetic description.

Proposition 1.14. *1) If $q \in \{0, \pm 1\}$ or $r = \pm 1$ then $T_{q,r}$ is trivial*

2) i) $T_{-q,r} = \bar{T}_{q,r}$

ii) Torus knots are invertible i.e. $T_{-q,-r} = T^r_{q,r}$ and $T_{q,r} \sim T_{-q,-r}$

Proposition 1.15. *Suppose that V is a trivial solid torus in S^3. Then $S^3 - int(V)$ is also a trivial solid torus.*

Proof. We can see S^3 as two 3-balls glued along their boundaries. Alter these two 3-balls so that they become solid cylinders W_1 and W_2. Now first, glue them on their tops, bases

and side surfaces respectively. Next, consider E_1, E_2 two cylinders with relatively small diameters respectively in W_1, W_2. We can think of them as fattened centres of the bigger cylinders. Now glue W_1 and W_2 along their boundaries as described above and hence E_1 and E_2 form trivial solid torus V in $S^3 = W_1 \cup W_2$.
Now to prove the proposition, we have to show that the solid obtained by extracting $int(V)$ is also a solid torus.
Create two rectangles in W_1 and W_2, and get a disk D obtained by these rectangles. Now rotating D around E_1 and E_2 which are glued, we get a solid torus by construction and moreover we get exactly $S^3 - int(V)$ $\square$

In particular, the circle m surrounding the core of $E_!$ is the meridian of V and is the longitude of the trivial solid torus $S^3 - int(V)$. Similarly for the longitude.

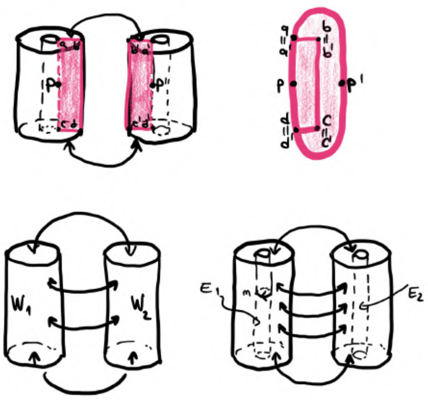


Figure 1.31

This proof is the main substance of the following proposition.

Proposition 1.16. *S^3 can be obtained by identifying (gluing) the surfaces i.e. tori of two trivial solid tori V_1 and V_2 in such a way that the meridian and the longitude of V_1 is glued to the longitude and meridian of V_2.*

Theorem 1.19.

$$T_{q,r} \cong T_{r,q}$$

Proof. Now we think of S^3 as gluing two solid tori in the previous way. If we think $T_{r,q}$ as lying on the torus T_1 it wraps its the meridian r times and the longitude q times. But then thinking of it on T_2, since the meridians and longitudes interchange we get the knot $T_{q,r}$. But since the knot itself has not changed, we are done. $\square$

The Seifert matrix of a torus knot

Suppose $q, r > 0$. While we introduced the Seifert matrix, we investigated on the example $T_{3,4}$. In general, the torus knots have Seifert matrices of a certain form.

Proposition 1.17. *Let $q, r > 0$. The Seifert matrix M of $T_{q,r}$ is S-equivalent to a $(r-1)\cdot(q-1)$ square matrix which can be divided into the following $(r-1)^2$ blocks:*

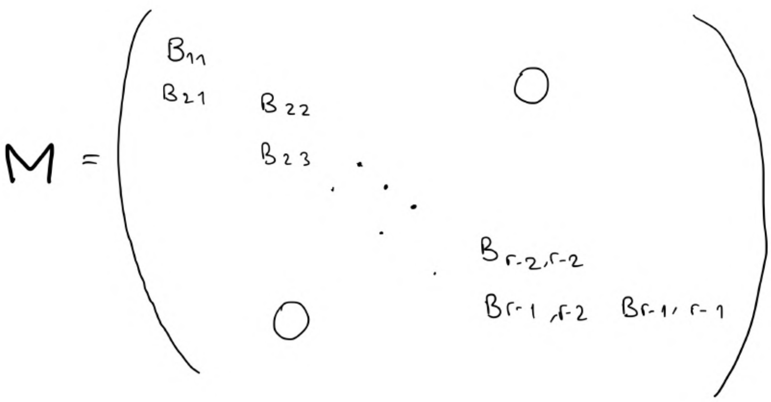


Figure 1.32

Such that

(1) Each block is a $(q-1)\cdot(q-1)$-matrix

(2) The diagonal blocks $B_{i,i}$ and the off-diagonal blocks $B_{i+1,i}$ are the only non zero matrices.

(3) $B_{i,i}$ is a $(q-1)\times(q-1)$ matrix and $B_{i,i}$ is given by:

$$B_{i,i} = \begin{pmatrix} -1 & & & & 0 \\ 1 & -1 & & & \\ & \ddots & \ddots & & \\ & & & -1 & \\ 0 & & & 1 & -1 \end{pmatrix}$$

Figure 1.33

(4) $B_{i+1,i} = -B_{1,1}$

Hence, we can use the above matrix to compute the Alexander polynomial of $T_{q,r}$.

Theorem 1.20. *Let $\Delta_{q,r}(t)$ denote the Alexander polynomial of $T_{q,r}$. Then*

$$t^{\frac{(q-1)(r-1)}{2}} \Delta_{q,r}(t) = \frac{(1-t)(1-t^{qr})}{(1-t^q)(1-t^r)}$$

Moreover, notice that the maximum degree of the Alexander polynomial $\Delta_{q,r}(t)$ is

$$qr + 1 - (q - r) - \frac{(q-1)(r-1)}{2} = \frac{(q-1)(r-1)}{2}$$

Here is the classification theorem

Theorem 1.21. *(1) If q or r is 0 or ± 1 then $T_{q,r}$ is the trivial knot*

(2) Suppose $r, q \neq 0, 1, -1$. Then

$$T_{q,r} \cong T_{p,s} \leftrightarrow \{q, r\} = \{p, s\} \quad \textit{or} \quad \{-p, -s\}$$

Invariants of torus knots

Theorem 1.22. *The signature of $T_{q,r}$ is not zero. Therefore $T_{q,r}$ is not amphicheiral.*

To actually determine the signature of a knot is quite complicated task, but to show it is not zero is relatively easier. We take the Seifert matrix M above and show that $\sigma(-(M + M^T))$ is positive considering the case where r is even or odd.

Theorem 1.23. *The following recurrence formula holds for the signature of the torus knot $T_{q,r}$:*

(I) Assume $2r < q$, then

(i) if $r = 0 \pmod 2$*, then* $\sigma(q,r) = \sigma(p-2r,r) + r^2 - 1$

(ii) if $r = 0 \pmod 2, \sigma(q,r) = \sigma(p-2r,r) + r^2$

(II) $\sigma(2r,r) = r^2 - 1$

(III) Let $r \leq q < 2r$

i if $r = 1 \pmod 2, \sigma(q,r) + \sigma(2r,q,r) = r^2 - 1$

ii if $r = 0 \pmod 2, \sigma(q,r) + \sigma(2r-q,r) = r^2 - 2$

(IV) $\sigma(q,r) = \sigma(r,q)$
$\sigma(q,1) = 0$
$\sigma(q,2) = q - 1$

Example 1.9. *We compute $\sigma(14,5)$:*

$$By\ (I) : \sigma(14,5) = \sigma(4,5) + 24 = \sigma(5,4) + 24$$
$$By\ (II) : \sigma(5,4) + \sigma(3,4) = 14$$

Also

$$\sigma(4,3) + \sigma(2,3) = 8$$
$$\sigma(4,3) = \sigma(3,4) = 6$$

Hence we get

$$\sigma(14,5) = 32$$

Theorem 1.24. *The genus of the torus knot $T_{q,r}$ is given by:*

$$g(T_{q,r}) = \frac{(q-1)(r-1)}{2}$$

Proof. Again, we take the Seifert matrix above. Now we have that

$$\begin{aligned} det(M) &= (detB_{1,1}) \cdots (detB_{r-1,r-1}) \\ &= [(-1)^{q-1}]^{r-1} \qquad\qquad \neq 0 \end{aligned}$$

Then, $g(T_{q,r})$ must be the maximum degree of the Alexander polynomial and hence it is $\frac{(q-1)(r-1)}{2}$ by 1.12. □

Finally, on the diagram we see that $(T_{q,r})$ has at exactly $|q|(|r|-1)$ crossing points. Moreover the crossing number $(T_{q,r})$ depends only on those diagrams by the following theorem.

Theorem 1.25. *The minimum number of crossings of $T_{q,r}$ is*

$$c(T_{q,r}) = min\{|q|(|r|-1), |r|(|q|-1)\}$$

But to determine the unknotting number of the torus knots is a difficult problem. We still have an easy inequality (exercice in [Mur96])

Claim 1.1. $u(T_{q,r}) \leq \frac{(q-1)(r-1)}{2}$

Theorem 1.26.

$$u(T_{q,r}) = \frac{(q-1)(r-1)}{2}$$

The above result will be recovered using the Rasmussen invariant s in the next chapter.

1.2 Algebraic background

In this second part of preliminaries, we set notations and give basic definitions and results on chain complexes, gradings, filtrations and spectral sequences that we will be using.

1.2.1 Chain complexes and their homologies

For our purposes, all complexes are taken over either $\mathbb{Q}$-vector spaces or $\mathbb{Z}$-modules.

Definition 1.36. *A chain complex C is a sequence $(C_i)_{i\in\mathbb{Z}}$ of $\mathbb{Q}$-vector spaces together with linear maps $\partial_i : C_i \to C_{i+1}$ such that $\partial_i \circ \partial_{i-1} = 0\ \forall i \in \mathbb{Z}$ ie. $Im(\partial_{i-1}) \subset Ker(\partial_i)$.*

Definition 1.37. *The homology $H_*(C)$ is the sequence $(H_i(C))_{i\in\mathbb{Z}}$ defined by*

$$H_i(C) := \frac{Ker(\partial_i)}{Im(\partial_{i-1})}$$

For $x \in Ker(\partial_i)$ we denote its homology class $[x]$ in $H_i(C)$.

Definition 1.38. *We say that a chain complex is acyclic if it has zero homology.*

Definition 1.39. *If C, D are two chain complexes we define the tensor product $C \otimes D$ as the complex defined by*

$$(C \otimes D)_i = \bigoplus_{k\in\mathbb{Z}} C_k \otimes D_{i-k}$$

$$d(x \otimes y) = d_C(x) \otimes y + (-1)^{|x|} x \otimes d_D(y)$$

where $|x|$ denotes the homological degree of $x \in C$.

Definition 1.40. *A decreasing chain complex D is a sequence $(D_i)_{i\in\mathbb{Z}}$ of $\mathbb{Q}$-vector space switch linear maps $\partial_i : C_{i+1} \to C_i$ such that $\partial_i \circ \partial_{i+1} = 0 \ \forall i \in \mathbb{Z}$*

To any chain complex one can associate a dual decreasing chain complex

$$\check{C}_i := Hom(C_i, \mathbb{Q}),$$
$$\check{\partial}_i(f) := f \circ \partial_i$$

Then the cohomology of $\check{C}$ is defined as

$$H^i(C) := \frac{Ker(\check{d}_i)}{Im(\check{d_{i+1}})}$$

Definition 1.41. *An exact sequence is a chain complex with homology equal to zero i.e. such that $Im(\partial_{i-1}) = Ker(\partial_i) \ \forall i \in \mathbb{Z}$.*

Definition 1.42. *For a chain complex C whose total rank $\sum_{i\in\mathbb{Z}} rk(C_i)$ is finite, we define its Euler characteristic by*

$$\chi(C) = \sum_{i\in\mathbb{Z}} (-1)^i rk(C_i)$$

Remark 1.7. *By the rank-nullity theorem one gets*

$$\chi(H_*(C)) = \chi(C)$$

where $H_*(C)$ is a chain complex with trivial maps.

1.2.2 Gradings

Definition 1.43. *We say that W is a graded $\mathbb{Z}-module$ if W admits a decomposition into direct sums*

$$W = \oplus_{i\in\mathbb{Z}} W_i$$

of $Z - modules$.
Similarly, we say that W admits a bi-grading if

$$W_i = \oplus_{j\in\mathbb{Z}} W_{i,j} \ \forall i \in \mathbb{Z}$$

Example 1.10. *For example if C is a chain complex, we can decompose its homology into a direct sum of vector spaces.*

$$H_*(C) = \oplus_{i\in\mathbb{Z}} H_i(C)$$

and i is called the homological grading.

Definition 1.44. *For any chain complex C and integer $k \in \mathbb{Z}$ we define the chain complex $C[k] := (\partial[k]_i : (C[k]_i \to C[k]_{i+1})_{i\in\mathbb{Z}}$ where*

$$C[k]_i := C_{i-k}$$
$$\partial[k]_i := (-1)^k \partial_{i-k}$$

obtained by shifting downwards the homological grading of C by k.

Remark 1.8. *We have in particular*

$$\chi(C[k]) = (-1)^k \chi(C)$$

Definition 1.45. *An internal grading on a chain complex C is a decomposition of $\mathbb{Z}$-modules*

$$C_i = \oplus_{j\in\mathbb{Z}} C^{i,j}$$

for each $i \in \mathbb{Z}$

Definition 1.46. *A chain map f is said graded of degree k if for every $j \in \mathbb{Z}$, $\partial_i : C_{*,j} \to C_{*,j+k}$. It is said graded if it is graded of degree 0 A graded chain complex is a chain complex over graded $\mathbb{Z}$-modules with graded differentials.*

Definition 1.47. *Let $V = \bigoplus_{i\in\mathbb{Z}} V_i$ be a graded abelian $\mathbb{Z}$ module. We define its dual $V*$ as a graded $\mathbb{Z}$ module by*

$$(V*)_i = Hom(V_{-i}, \mathbb{Z})$$

Definition 1.48. *Let C be a graded complex over $\mathbb{Z}$-modules. Then define its dual $C*$ as the complex defined by*

$$(C*)^i = (C_{-i})*$$
$$(d*)^i = (d^{i+1})*$$

Remark 1.9. *Note that if C is graded, then the boundary maps endow $C_{*,j}$ with a chain* *complex structure, for every $j \in \mathbb{Z}$ and hence C splits into a direct sum of chain complexes*

$$C = \oplus_{j\in\mathbb{Z}} C_{*,j}$$

Definition 1.49. *For any chain complex C given with an integral grading and for any $k \in \mathbb{Z}$ we define the chain complex*

$$C\{k\} := \oplus_{j\in\mathbb{Z}} C\{k\}_{.,j}$$

where $C\{k\}_{i,j} = C_{i,j-k}$ $\forall i,j \in \mathbb{Z}$. In other terms, we shift the integral grading by k and again, we have a bigrading on the complex $C\{k\}$.

Remark 1.10. *So in our conventions if C is a bigraded complex we have,*

$$C[k]\{p\}_{i,j} = C_{i-k,j-p}$$

We can view $C_{*,*}$ lying on a lattice as below, viewing the homological degree gr_h as the vertical one and the integral degree gr_q as the horizontal one. In that case shifting the homological grading by i means moving everything up by k and shifting the internal grading by p means moving everything right

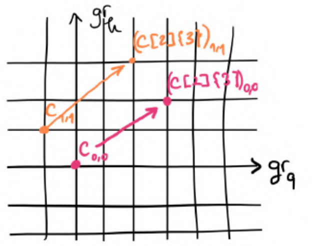


Figure 1.34: Bigrading shiftings

Definition 1.50. *The graded (quantum) dimension of a graded vector space $W = \oplus_{i\in\mathbb{Z}} W_i$ is the polynomial*

$$\text{qdim}(W) = \sum_{i\in\mathbb{Z}} q^i \dim(W_i)$$

It satisfies

- $\mathrm{qdim}(W \otimes W') = \mathrm{qdim}(W)\mathrm{qdim}(W')$
- $\mathrm{qdim}(W \oplus W') = \mathrm{qdim}(W) + \mathrm{qdim}(W')$
- $\mathrm{qdim}(W\{k\}) = q^k \mathrm{qdim}(W)$

Definition 1.51. *Let C be a graded chain complex with finite total rank. Then the graded Euler characteristic of C is defined as*

$$\begin{aligned}\chi_{gr}(C)(q) &= \sum_{j\in\mathbb{Z}} \chi(C_{*,j})q^j \\ &= \sum_{i,j\in\mathbb{Z}} (-1)^i rk(C_{i,j})q^j \quad \in \mathbb{Z}[q^{\pm 1}]\end{aligned}$$

Given a graded complex, we associate to the homology group the Poincaré polynomial where the dimensions of groups at different gradings appear as coefficients and which can be very useful for the computations. This point of view might sometimes simplify the computation of homology group.

Definition 1.52. *Let C be a graded complex of finitely generated-abelian groups and H its homology. We define its Poincaré polynomial by*

$$P_H(t,q) = \sum_{i\in\mathbb{Z}} t^i \mathrm{qdim}(H_{i,j})$$

Remark 1.11. *In particular one has $\chi(C_{*,*}) = P(-1,q)$*

Chain maps and their cones

For our purposes we can consider our complexes over $\mathbb{Z}$-modules or $\mathbb{Q}$-vector spaces.

Definition 1.53. *A chain map $f : C^1 \to C^2$ between two chain complexes C^1 and C^2 is a sequence of maps $(f_i : C^1_i \to C^2_i)_{i\in\mathbb{Z}}$ such that the below diagram commutes:*

$$\begin{tikzcd} C_i^1 \arrow[r, "f_i"] \arrow[d, "\partial_i^1"'] & C_i^2 \arrow[d, "\partial_i^2"] \\ C_{i+1}^1 \arrow[r, "f_{i+1}"'] & C_{i+1}^2 \end{tikzcd}$$

Figure 1.35

Definition 1.54. *Let $f : V \to W$ be a map between graded $\mathbb{Z}$-modules. Then f is said graded of degree d if $f(V_i) \in W_{i+d}$*

Similarly for a map between two graded chain complexes, f is graded of degree d if $f_i(C_i^1) \subset C_{i+d}^2$ $\forall i \in \mathbb{Z}$

Example 1.11. *For example the differential map induced on homology is graded of degree 1, and the one in the homology of cohomology is of degree -1.*

Remark 1.12. *If $f : C^1 \to C^2$ is a graded map of degree d, then $f : C^1\{m_1\} \to C^2\{m_2\}$ is graded of degree $d - m_1 + m_2$*

Proposition 1.18. *A chain map $f : C^1 \to C^2$ induces a well defined chain map $f^* : H_*(C^1) \to H_*(C^2)$ at the level of homology.*

Definition 1.55. *Let $F, G : C_1 \to C_2$ are two chain maps. We say that they are chain homotopic if there exists chain maps h_i such that diagram below commutes for every i*

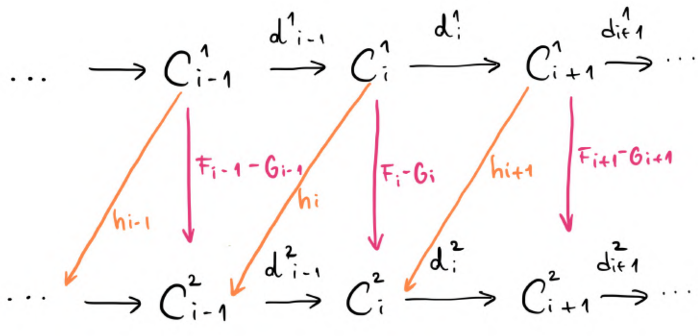


Figure 1.36

Definition 1.56. *A chain map $f : C_1 \to C_2$ is said to be null-homotopic if it is homotopic to the 0 chain map.*

Definition 1.57. *A complex (C, d) is said to be contractible if the identity chain map id_C is null homotopic in other terms if there exists a map $s : C_n \to C_{n-1}$ such that $id_C = sd + ds$.*

Proposition 1.19. *A contractible complex is acyclic.*

Proof. It is easy to see that if $x \in Ker(d)$ then

$$x = (sd + ds)(x) = d(s(x)) \in Im(d)$$

□

Let A, B, C be complexes over finite dimensional vector spaces:

$$A : \left(\cdots A_i \xrightarrow{d_i} A_{i+1} \cdots \right)$$
$$B : \left(\cdots B_i \xrightarrow{d_i} B_{i+1} \cdots \right)$$
$$C : \left(\cdots C_i \xrightarrow{d_i} C_{i+1} \cdots \right)$$

Let

$$0 \to A \xrightarrow{f} B \to gC \to 0$$

be a short exact sequence of complexes. This means that for every i we have short a exact sequence

$$0 \xrightarrow{f_i} A_i \xrightarrow{g_i} C_i \to 0$$

This induces a long exact sequence on homology:

$$\cdots \to H_i(A) \xrightarrow{f_i^*} H_i(B) \xrightarrow{g_i^*} H_i(C) \xrightarrow{\partial_i} H_{i+1}(A) \to \cdots$$

which we denote as a long exact *triangle*

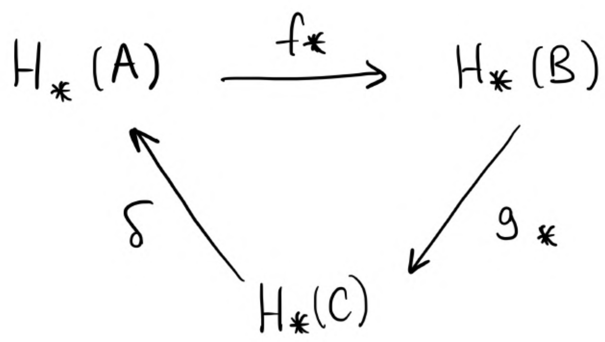


Figure 1.37

Lemma 1.1. *Let $A \to B \to C \to A$ be an exact triangle which splits, in other terms such that we have an isomorphism $B \cong A \oplus C$ such that the below diagram commutes*

$$\begin{tikzcd}
A \arrow[r] \arrow[d, "id"] & B \arrow[r] \arrow[d, "\wr"] & C \arrow[d, "id"] \\
A \arrow[r] & A \oplus C \arrow[r] & C
\end{tikzcd}$$

Figure 1.38

Then the morphism $C \to A[1]$ vanishes

Proof. First, by the triangle we get a long exact sequence

$$\cdots \to A \to B \to C \to A[1] \to \cdots$$

Now the splitting of the exact sequence gives a morphism $C \to B$ such that

$$C \to B \to C$$

is the identity morphism id_C. We decompose the morphism $C \to A[1]$ as

$$C \to B \to C \to A[1]$$

By the exactness, $B \to C \to A[1]$ vanishes. □

Definition 1.58. *If $f : C^1 \to C^2$ is a chain map, then $Cone(f)$ is the chain complex defined as:*

$$C_i^2 \oplus C_i^1[-1]$$

with boundary maps ∂_i defined by:

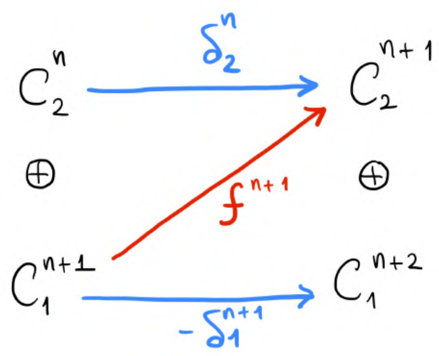


Figure 1.39

Remark 1.13. *If the map f is graded then $Cone(f)$ is graded.*

Lemma 1.2. *For any graded chain map f we have*

$$\chi_{gr}(Cone(f)) = \chi_{gr}(C^1) - \chi_{gr}(C^2)$$

Proposition 1.20. *For any chain map $f : C^1 \to C^2$ there is an exact sequence*

$$\cdots \to H_{i-1}(C_i^2) \xrightarrow{\iota^*} H_i(Cone(f)) \xrightarrow{\pi^*} H_i(C_i^1) \xrightarrow{f^*} H_i(C_i^2) \to \cdots$$

1.2.3 Filtrations and spectral sequences

For our purposes, we consider all our filtrations over $\mathbb{Q}$-vector spaces.

Definition 1.59. *Let V be a $\mathbb{Q}$-vector space. A finite length decreasing filtration F^* on V is a family of sub-vector spaces $(F^k(V))_{k\in\mathbb{Z}}$ such that*

$$F^k(V) = 0 \ \forall k \geq n$$
$$F^k(V) = V \ \forall k \leq 0$$

$$\{0\} = F^n(V) \subset F^{n-1}(V) \subset \cdots F^1(V) \subset F^0(V) = V$$

One can view it as below

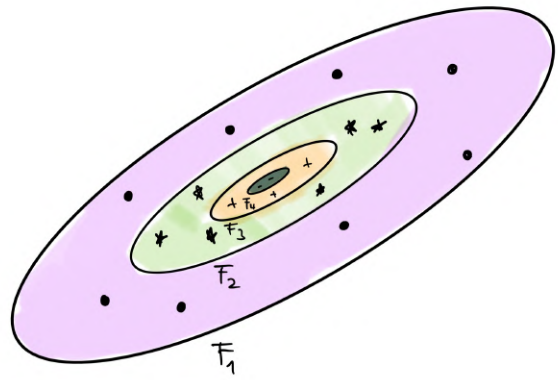


Figure 1.40

Definition 1.60. *Let V, W be two filtered vector spaces. A linear map $f : V \to W$ is said to be filtered of degree k if $\forall i \in \mathbb{Z}$*

$$f(F^i(V)) \subset F^{i+k}(W)$$

A filtered map of degree 0 is called filtered.

Definition 1.61. *A filtered chain complex is a chain complex (C, d) where each chain space is a filtered vector space and where the differenials are filtered. In particular this induces a filtration on homology groups by*

$$F^i(H_n(C,d)) := Ker(d_i) \cap F^i(C_n)/Im(d_n) \cap F^i(C_n)$$

Moreover, one can collapse a filtered vector space V into its associated graded vector space.

Definition 1.62. *Let V be a filtered vector space. We define its associated graded vector space $E_0^*(V)$ by :*

$$E_0^P(V) = F^p(V)/F^{p+1}(V) \forall p \in \mathbb{Z}$$

Remark 1.14. *Observe that the associated graded vector space is non trivial only in degrees $0 < k < n$.*

Remark 1.15. *In particular we get a grading on V since we have*

$$V = \oplus E_0^p(V)$$

Also, we get short exact sequences:

$$0 \to F^n(V) \xrightarrow{\sim} E_0^n(V) \to 0$$
$$0 \to F^n(V) \to F^{n-1}(V) \to E_0^{n-1}(V) \to 0$$
$$0 \to F^{n-1}(V) \to F^{n-2}(V) \to E_0^{n-2}(V) \to 0$$

In particular, if H^* is a graded $\mathbb{Q}$-vector space and if H^* is filtered, then we can examine the filtration at each homological degree, letting

$$F^p H^n = F^p H^* \cap H^n$$

This makes the associated graded vector space into a a bigraded vector space:

$$E_0^{p,q}(H^* F) = \frac{F^p H^{p+q}}{F^{p+1} H^{p+q}}$$

1.2.4 Spectral sequences

In a general setting where when one has a graded $\mathbb{Q}$-vector space H^* and where H^* seems in general difficult to obtain, one can use a filtration on H^* to approximate it via what we call a spectral sequence.Here we only work over $\mathbb{Q}$-vector spaces.

Definition 1.63. *A spectral sequence is a sequence of differential bigraded vector spaces, that is $\forall r \geq 1,\ p, q \geq 0$ there is $E_r^{p,q}$ and each bigraded vector space $E_r^{*,*}$ is equipped with a map $d_r : E_r^{*,*} \to E_r^{*,*}$*

which is a differential i.e. $d_r \circ d_r = 0$ of bidegree $(s(r), t(r))$:

$$d_r : E_r^{p,q} \to E_r^{p+s(r),q+t(r)}$$

and such that

$$\forall r \geq 1, E_{r+1}^{*,*} \cong H(E_r^{*,*}, d_r)$$

The term $E_r^{*,*}$ is usually called the r^{th} page of the spectral sequence. Hence a spectral sequence can be pictured as a sequence of pages where every time we turn the page, we take homology.

:

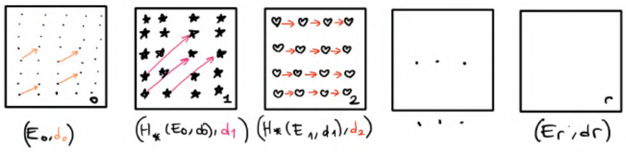


Figure 1.41

With the above definitions consider $E_r^{p,q}$ for $r \gg 0$. If the differentials become trivial say for $r \geq N$,we have $E_{N+k}^{p,q} = E_N^{p,q} \forall k \geq 0$

Denote $E_\infty^{p,q}$ this vector space at which the spectral sequence collapses.

Definition 1.64. *A spectral sequence $\{E_r^{*,*}, d_r\}$ of vector spaces is said to converge to a graded vector space H^* if H^* has a filtration F^* and if*

$$E_\infty^{p,q} \cong \ F^p H^{p+q} / F^{p+1} H^{p+q} = E_0^{p,q}(H^*)$$

It follows that H^* is *approximated* if we can find a spectral sequence converging to H^*.

Definition 1.65. *We say that a spectral sequence $E_{p,q}^r$ is bounded if $\forall n, r \in \mathbb{Z}, E_{k,n-k}^r$ has finitely many non zero terms.*

Proposition 1.21. *A bounded spectral sequence has a limit page.*

Filtration spectral sequence

One important example of obtaining a spectral sequence is that of the filtration spectral sequence.

Definition 1.66 (from [Zha25]). *Consider a filtered complex (C, d) of $\mathbb{Q}$-vector spaces with filtration $\{F_j C\}$ and where d decomposes as a sum of filtered maps of degree i*

$$d = d0 + d^1 + \cdots + d^n$$

. Then the filtration spectral sequence for (C, d) is recursively defined as follows:

- $(E^0, d_0) := (C, d^0)$
- (E^1, d_1)

  - E^1 *is the homology of the* (E^0, d_0)
  - d_1 *is the graded degree 1 piece of the induced* d_* *on homology*

- $(E^r, d_r), \forall r \in \mathbb{Z}$*:*

  -E^r *is the homology of* (E^{r-1}, d_{r-1})
  -d_r *is the degree-r graded piece of* d_* *induced on rth homology*

- *For all* $r, p, q; E^r_{p,q}$ *has quantum grading* q *and homological grading is defined by* $p + q$*. The differential* d_r *on the page* E^r *is of bidegree* $(-1 + r, -r)$ *for the grading* (gr_h, gr_f)*.*

Remark 1.16. *In particular since* $p + q \mapsto (p + 1 - r) + (q + r) = p + q + 1$*, on each page the homological gradings are getting shifted by* 1*.*

Remark 1.17. *Attention! In this spectral sequence, the differential* d_r *on the rth page is NOT directly induced by the degree-r piece* d^r *of* d*! Indeed this holds only for the differential on the* 0*th page!*

Below is an illustation which gives the idea of what the filtration spectral sequence might look like

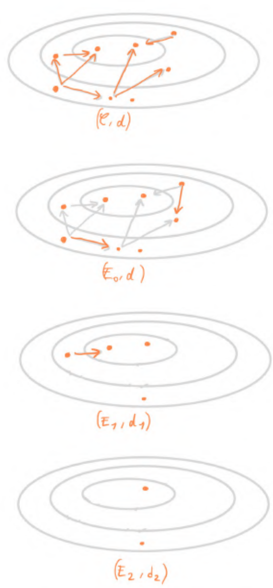


Figure 1.42

Theorem 1.27. *If the filtration spectral sequence of a filtered complex has a limit page, the limit page is isomorphic to the chain homology of the original filtered complex.*

Proposition 1.22. *Let $f : C \to C'$ be a filtered map of filtered complexes. Then f induces a map of spectral sequences (on the associated filtration spectral sequences) $f^n : E^n \to E'^n$ and if f^m is an isomorphism, $\forall n \geq m, f^n$ is an isomorphism.*

Lemma 1.3. *If two filtered complexes are dual, so are their corresponding filtration spectral sequences: $E^r_{p,q} \cong E'^r_{-p,-q}$*

TQFT's and Frobenius algebras

Here we give a brief description of TQFT's (Topological quantum field theory) and their relations to algebraic structures called Frobenius algebras.

Definition 1.67. *Let Σ, Σ' be d-dimensional smooth, closed, oriented manifolds. A cobordism M from Σ to Σ' is a $d+1$-dimensional smooth oriented manifolds such that $\partial M = \overline{\Sigma} \sqcup \Sigma'$.*

In all what follows in this work, our cobordisms are oriented from top to the bottom of the page.

A TQFT is a functor satisfying the following axioms.

Let $d \in \mathbb{N}$. A $(d+1) - TQFT$ is defined over a ground ring Λ. In these notes think of Λ as $\mathbb{Z}$ or $\mathbb{Q}$.

(A) $Z(\Sigma)$ is a Λ-module for each Σ closed, oriented smooth d-dimensional manifold.

(B) $Z(M) \in Z(\partial M)$ for $(d+1)$-dimensional manifold with boundary such that:

  (1) Z is functorial with orientation preserving diffeomorphism of Σ and M. If $f : \Sigma \to \Sigma'$ is such a diffeomorphism then $Z(f)$ is an isomorphism, $Z(fg) = Z(f)Z(g)$ and if f extends to $f : M \to M'$ where $\partial M = \Sigma$ and$\partial M' = \Sigma'$ then $Z(f)(Z(M)) = Z(M')$.

  (2) Z is involutive: $Z(\Sigma^*) = Z(\Sigma)^*$ where Σ^* is the manifolds with reversed orienation.

  (3) Z is multiplicative:

    (3a) $Z(\Sigma_1 \sqcup \Sigma_2) = Z(\Sigma_1) \otimes Z(\Sigma_2)$

    (3b) if $\partial M_1 = \Sigma_1 \sqcup \Sigma_3$ and $\partial M_2 = \Sigma_2 \sqcup \Sigma_3^*$ with $M = M_1 \cup_{\Sigma_3} M_2$ then $Z(M) = \langle Z(M_1), Z(M_2) \rangle$

(3c) $Z(M) = Z(M_1) \otimes Z(M_2)$

where $\langle ., . \rangle$ stands for the map $Z(\Sigma_1) \otimes Z(\Sigma_3) \otimes Z(\Sigma_2) \to Z(\Sigma_1) \otimes Z(\Sigma_2)$

(4a) $Z(\emptyset) = \Lambda$ for the d-dimensional $\emptyset$

(4b) $Z(\emptyset) = 1$ for the $d+1$-dimensional $\emptyset$

Some remarks on these axioms:

Remark 1.18. *Note that (2) is a strong axiom since it gives a computability of $Z(M)$ by cutting M in halfway along any Σ_3.*

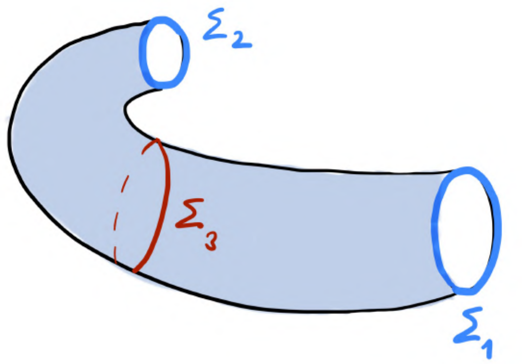


Figure 1.43

Remark 1.19. *If $\partial M = \Sigma_1 \cup \Sigma_0^*$, then we have $Z(M) \in Z(\Sigma_1) \otimes Z(\Sigma_0)^* = Hom(Z(\Sigma_0), \Sigma_1)$. Hence if M is a cobordism $M : \Sigma_0 \to \Sigma_1$ then $Z(M) : Z(\Sigma_0) \to Z(\Sigma_1)$. So we can view any cobordism M from Σ_0 to Σ_1 as an induced linear transformation. We add the following axiom*

(4c) $Z(\Sigma \times I)$ is the identity $Z(\Sigma) \to Z(\Sigma')$

Let us see some consequences of these axioms:

- Apply axiom (1) with $M = M' = \Sigma \times I$. Let $F : M \to M'$ be a homotopy f_t of maps $\Sigma \to \Sigma'$. We deduce the homotopy invariance of $Z(f) : Z(\Sigma) \to Z(\Sigma)$. In other terms, if $f : \Sigma \to \Sigma$ is a diffeomorphism isotopic to id, then this isotopy induces a map $F : \Sigma \times [0, 1] \to \Sigma \times [0, 1]$. Then $Z(f) = Z(\Sigma \otimes I) = Id$.

- If M is closed so if $\partial M = \emptyset$ then $Z(M) \in Z(\emptyset) = \Lambda$ is constant. Hence, the theory produces invariants of $d+1$-dimensional closed manifolds. Moreoever computations of these invariants can be done by decomposing $M = M_1 \cup M_2$.

In our setting of Khovanov homology, we will be only concerned by $(1+1)$ TQFT's over $\mathbb{Z}$ or $\mathbb{Q}$. So we can think about a $(1+1)$ TQFT as a functor Z as above from the category of finite collections of circles and (smooth) cobordisms between them to the category of finitely generated $\mathbb{Z}$- or $\mathbb{Q}$- modules such that $Z(Y \sqcup Y') = Z(Y) \otimes Z(Y')$, $Z(Y^*) = Z(Y)^*$ and $Z(Y \times I) = Id_{Z(Y)}$

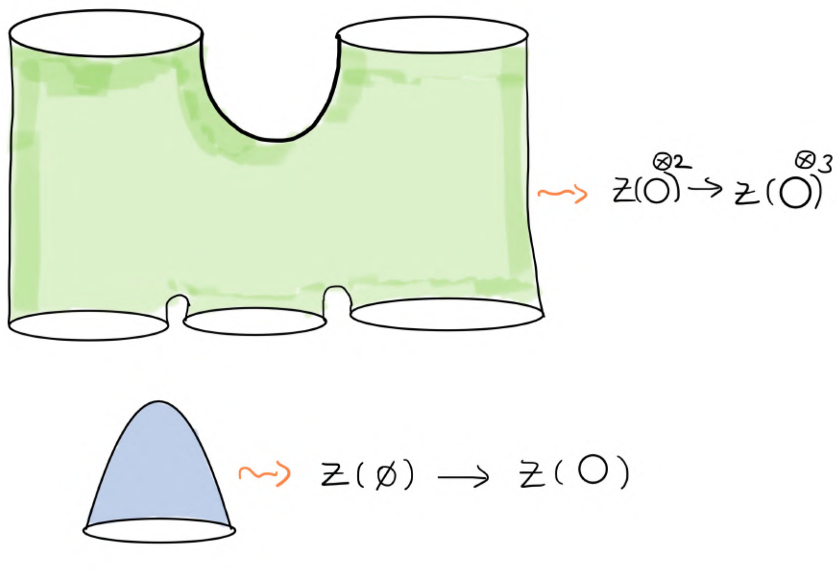


Figure 1.44

Definition 1.68. *A co-algebra is a triple (C, Δ, ε) where C is a $\mathbb{Q}$-vector space and $\Delta : C \to C \otimes C, \varepsilon : C \otimes C \to C$ are linear maps satisfying*

$$(\Delta \otimes id) \circ \Delta = (id \otimes \Delta) \circ \Delta$$
$$(\varepsilon \otimes id) \circ \Delta = id = (id \otimes \varepsilon) \circ \Delta$$

Definition 1.69. *A Frobenius system is the data $(R, A, \varepsilon, \Delta)$ consisting of:*

- *A commutative ring R*
- *A R-algebra A with a map $\iota : R \to A$ such that $\iota(1) = 1$*
- *A multiplication map $m : A \otimes_R A \to A$*
- *A comultiplication map: $\Delta : A \to A \otimes_R A$*
- *An R-module counit map $\varepsilon : A \to R$ such that $(\varepsilon \otimes id) \circ \Delta = id$*

And the algebra A is called a Frobenius algebra, it is both an algebra and coalgebra.

The following relations hold:

$$
\begin{aligned}
&(id \otimes m) \circ (\Delta \otimes id) = \Delta \circ m = (m \otimes id) \circ (id \otimes \Delta) \\
&m(m \otimes id) = m(id \otimes m) \\
&(\Delta \otimes id)\Delta = (id \otimes \Delta)\Delta \\
&m(\iota \otimes id) = id = m(id \otimes \iota) \\
&(\varepsilon \otimes id)\Delta = id = (id \otimes \varepsilon)\Delta \\
&\Delta \circ m = (m \otimes)(id \otimes \tau \otimes id)(\Delta \otimes \Delta)
\end{aligned}
$$

There is a strong relationship between TQFT's and Frobenius algebras. For example let A be the rank 2-Frobenius extension of the ground ring $\mathbb{Z}$:

$$A = \mathbb{Z}[X]/X^2$$

This yields to a $(1+1)$ dimensional TQFT as we shall see in the next chapter.

Chapter 2

Khovanov homology and the Rasmussen invariant

This chapter is organised as follows: first we define the Khovanov homology for links and show that it categorifies the Jones polynomial. We mention some general properties that will be useful to us. Next, we consider another homological invariant called Lee-Khovanov homology and study the spectral sequence which connects these two homology theories to each other. Finally we define the Rasmussen invariant, consider basic properties and prove our main theorem: the Milnor conjecture. Our references for this section are [Kho00], [Ras10a], [Lee02], [Zha25], [Ras10b], [Tur06].

2.1 Khovanov homology

Khovanov homology was defined by Mikhail Khovanov in [Kho00] at the end of the $20th$ century, and appears as a categorification of the Jones polynomial. It is a bigraded $\mathbb{Z}$-module which is a link invariant (up to isotopy), and its Euler characteristic is the unnormalized Jones polynomial of the link.

2.1.1 Categorification of the Jones polynomial

Recall that the Jones polynomial is an isotopy invariant of links in S^3 that takes values in $\mathbb{Z}[q^{\pm 1}]$. It satisfies the following skein condition:

$$q^2 J(\diagup\!\!\!\!\diagdown) - q^{-2} J(\diagdown\!\!\!\!\diagup) = (q - q^{-1}) J(\asymp)$$

In particular, this implies

$$J(L \cup \bigcirc) = (q + q^{-1}) J(L)$$

So, adding the unknot to a link multiplies the Jones polynomial by $(q + q^{-1})$. We normalize by imposing:

$$J(\bigcirc) = (q + q^{-1})$$
$$J(\emptyset) = 1$$

The concept of categorification

Following the discussion in [AK08], let us try to get a feeling for the concept of categorification.

We consider a simple case: lifting natural numbers to vector spaces or to free abelian groups.
In the opposite direction (decategorification), we associate to a finite dimensional vector space its dimension and to a finitely-generated abelian group its rank.
Operations on vector spaces/groups mirror those on the natural numbers:

Direct sum $\rightsquigarrow$ sum of numbers
Tensor product $\rightsquigarrow$ multiplication of numbers

Let n, m be natural numbers and V, W finite dimensional vector spaces with dimensions respectively n, m. Then, one has an informal correspondence

$$\begin{aligned} n &\leftrightsquigarrow \mathbb{Z}^n \\ n + m &\leftrightsquigarrow V \oplus W \\ nm &\leftrightsquigarrow V \otimes W \end{aligned}$$

A very natural question to ask is:

What about $n - m$?

Now, we need to step beyond and consider complexes of vector spaces or abelian groups. The analogue of the dimension/rank will be the *Euler characteristic* of the complex.
Then, $n - m$ will correspond to the cone $Cone(d)$ where

$$0 \to V \xrightarrow{d} W \to 0$$

Indeed, consider

$$0 \to V \to 0$$
$$0 \to W \to 0$$

where V, W lie in homological degree 0 are complexes of Euler characteristic m, n respectively. Then for any linear map $f : V \to W$ the complexe $Cone(f)$ has Euler characteristic $n - m$.

$$\begin{aligned} n &\leftrightsquigarrow 0 \to V \to 0 \\ m &\leftrightsquigarrow 0 \to W \to 0 \\ n - m &\leftrightsquigarrow Cone(f) \end{aligned}$$

An example of categorification is passing from the Euler characteristic $\chi(X)$ of a topological space X to its homology groups $H_*(X, \mathbb{Z})$. We recover the Euler characteristic by:

$$\chi(X) = \sum_i (-1)^i rk(H_i(X, \mathbb{Z}))$$

For example singular homology categorifies the Euler characteristic. Indeed if X is a topological space admitting a finite CW-decomposition then

$$\chi(H_*(X)) = \sum_{i \in \mathbb{Z}} (-1)^i rk(H_i(X)) = \chi(X)$$

Here are the benefits of the (singular) homology compared to the Euler characteristic:

- The invariant is a graded abelian group, encoding more information than an integer.
- Homology extends to a functor from the category of the topological spaces and continuous maps up to homotopy to the category of graded abelian groups and grade-preserving homomorphisms. So, it provides information about continuous maps, where χ does not.
- Homology groups (singular) are defined for any topological space, whereas the Euler characteristic is only defined for those admitting a finite CW decomposition.

Categorification of the Jones polynomial

The motivation behind the definition of the Khovanov homology [Kho00] is the categorifictaion of the Jones polynomial.

Let L be a link.

Goal: Categorify the (unnormalized) Jones polynomial

$$J(L) = \sum_j a_j(L) q^j \in \mathbb{Z}[q^{\pm 1}]$$

Since the coefficients are integers, we will realize each coefficient as the Euler characteristic of a $\mathbb{Z}$-graded homology theory. Then considering all the coefficients, we will get a bigraded homology groups associated to the link L:

$$H(L) = \bigoplus_{i,j \in \mathbb{Z}} H^{i,j}(L)$$

So that

$$a_j(L) = \sum_i (-1)^i rk(H^{i,j}(L))$$

hence

$$J(L) = \sum_{i,j} (-1)^i rk(H^{i,j}(L) q^j)$$

This homology will be constructed by lifting the Kauffman bracket formula for the Jones polynomial to complexes using a $(1+1)$ TQFT.
To an oriented diagram D of an oriented link we will associate a complex $C(D)$ of free abelian graded groups with a grading preserving differential d:

$$C(D) : \cdots \to C^{i-1}(D) \xrightarrow{d} C^i(D) \xrightarrow{d} C^{i+1}(D) \to \cdots$$

where

$$C^i(D) := \bigoplus_{j \in \mathbb{Z}} C^{i,j}(D)$$

In particular, for a given degree j, the complex restricts to a complex of free abelian groups:

$$C^j(D) : \cdots \to C^{i-1,j}(D) \xrightarrow{d} C^{i,j}(D) \xrightarrow{d} C^{i+1,j}(D) \to \cdots$$

with $a_j(L) = \chi(C^j(D))$

We think of the groups $C^{i,j}(D)$ as vertices of the lattice $\mathbb{Z}^2$ and the differentials as lying on the edges of the lattice:

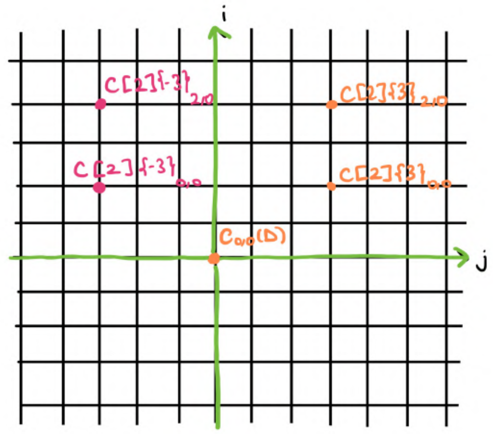


Figure 2.1

Recall that the grading shiftings yield to new complexes, hence to a new picture on the lattice where:

- The grading shift $\{1\}$ moves everything 1 step right (shifting on the integral grading)
- The grading shift $[1]$ moves everything one step up (shifting on the homological grading)

ie if $C^{*,*}$ is a bigraded complex then we obtain a bigradded complex where

$$C[k_1]\{k_2\}^{i,j} = C^{i-k_1,j-k_2}$$

Recall that if Y_1, Y_2 are two closed 1-dimensional manifolds, an $(1+1)$-dimensional cobordism from Y_1 to Y_2 is a compact oriented surface W such that $\partial W = Y_1^* \sqcup Y_2$. We say that two such cobordisms are equivalent if there exists a homeomorphism between them such that restrictions on boundaries is the identity. We denote the category of $(1+1)$-dimensional cobordisms by Cob^3 which has for objects closed oriented 1-dimensional manifolds and equivalence classes of cobordisms between them. The composition between $W_1 : Y_1 \to Y_2$ and $W_3 : Y_2 \to Y_3$ is given by the cobordism $W2 \circ W_1 := W_1 \sqcup_{Y_2} W_2 : W_1 \to W_3$. In other terms, we glue one cobordism atop the other as shown below:

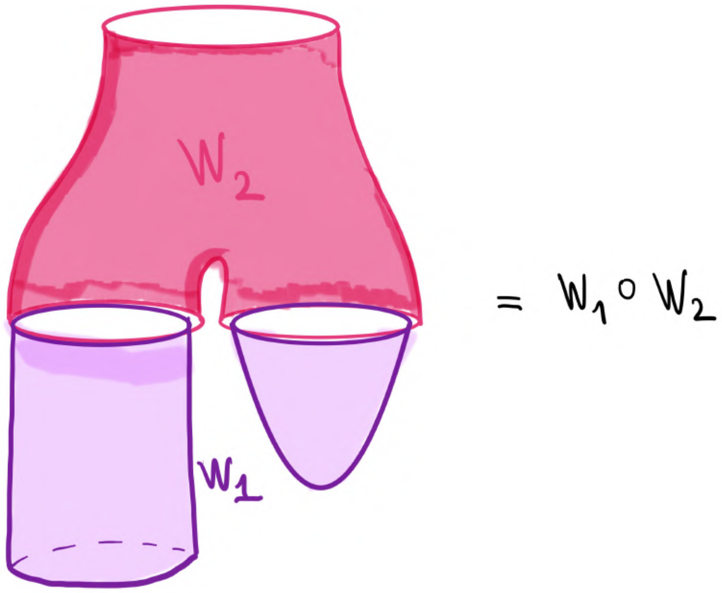


Figure 2.2: Composition in Cob^3

Let D be a link diagram and forget about the orientation for the moment. Consider the cube of resolutions that we mentioned at the end of the section concerning the Jones polynomial.

Recall that the cube of resolution of an unoriented (oriented) link diagram D with n crossings is an oriented n dimensional cube having for each vertex a resolution $r \in \{0,1\}^n$ of D. Two vertices are connected by an edge if they differ by a crossing resolution at the ith coordinate for $0 \leq i \leq n$. An edge between r, r' is oriented from the resolution having $r_i = 0$ to that having $r'_i = 1$.

Now we will view this cube of resolutions in the category of cobordisms by the following transformation.

Associate a sign to an edge e

If r is a resolution define its Hamming weight by $|r| = \#\{i \mid r_i = 1\}$, associate to it the resolved diagram D_r which is a collection of $|D_r|$ many disjoint circles. Endow it with the canonical orientation defined by giving the kth circle C_k the $(-1)^{n_k}$ times the counter-clockwise orientation where n_i is the number of circles separating C_k from the infinity in $\mathbb{R}^2$ as illustrated on an example below.

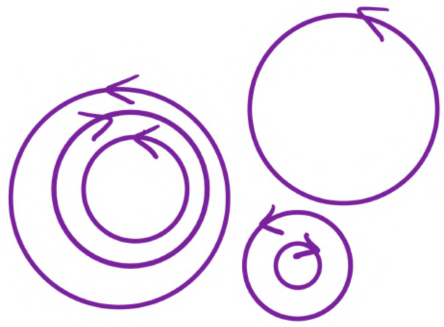


Figure 2.3

If r and r' are two resolutions which differ by a crossing resolution and if e is the oriented edge between them then define the cobordism S_e by taking the product cobordism outside the neighbourhood where no change occurs and place a saddle cobordism in the neighbourhood where there is a change, shown as below

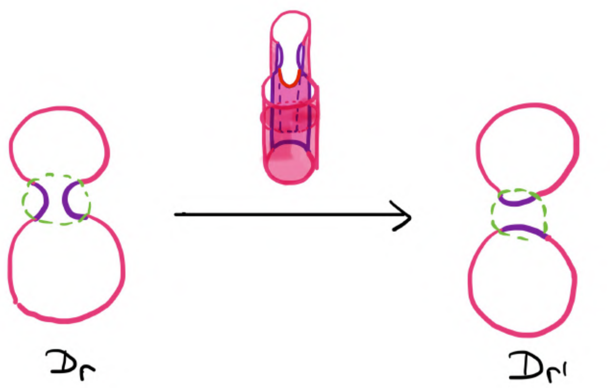


Figure 2.4

This is equivalent to taking the product cobordism $D_r \times I$ and then attaching a one handle whose core is the red curve as shown in the figure above.
This transforms the cube of resolutions to the cube of resolved diagrams, replacing the vertex belonging to the resolution r by the canonically oriented D_r and replacing each oriented edge e by the cobordism S_e.

Lemma 2.1. *Each 2-dimensional face of the cube commutes*

Proof. 1-handles can be added in any order without changing the homeomorphism type of the surface, as illustrated by the figures below. We consider each possible face in the cube:

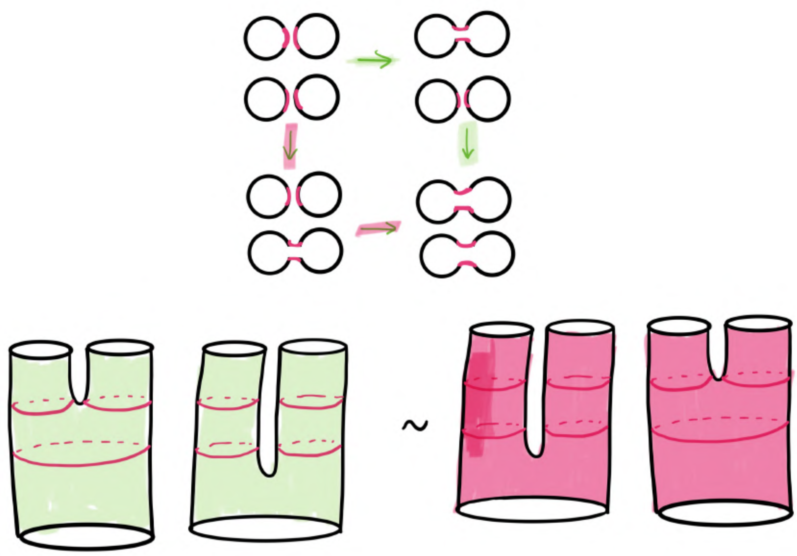


Figure 2.5

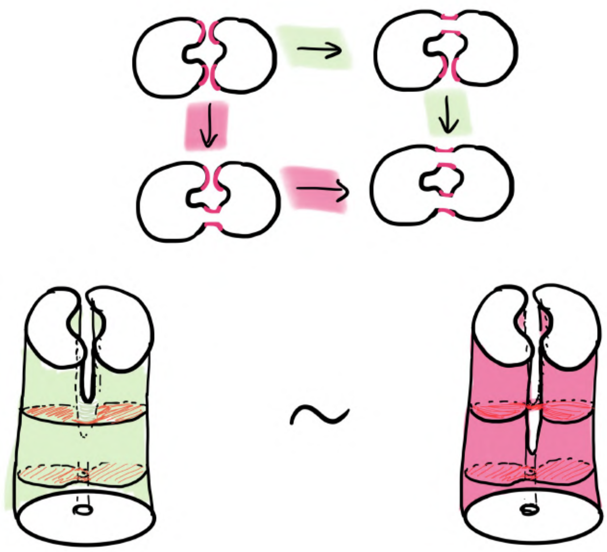


Figure 2.6

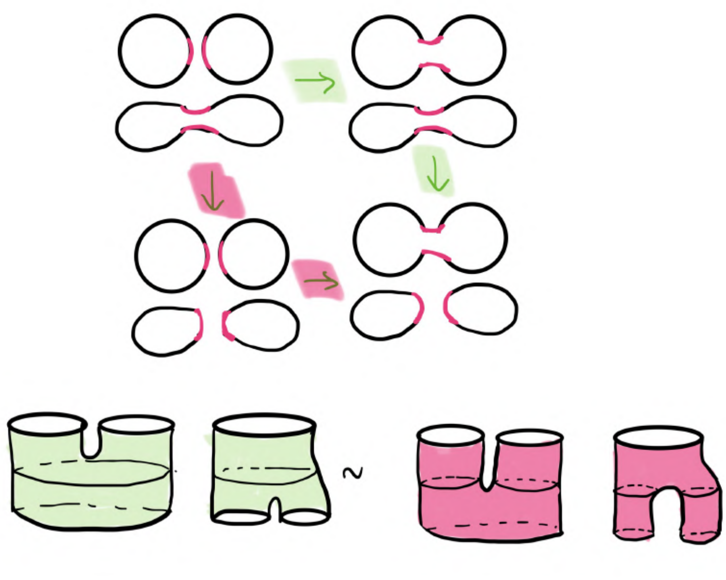


Figure 2.7

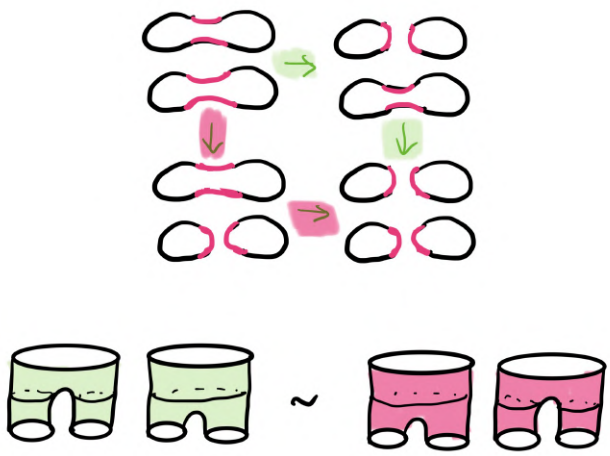


Figure 2.8

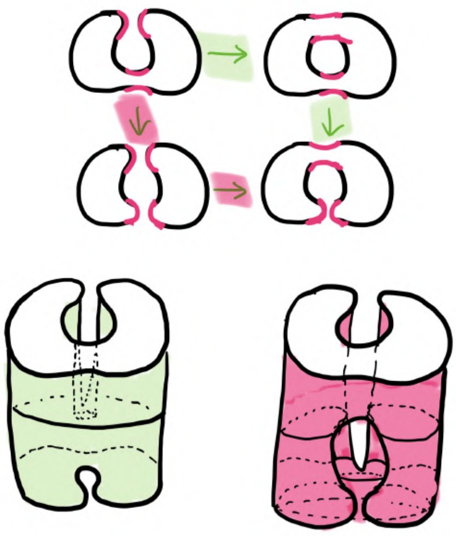


Figure 2.9

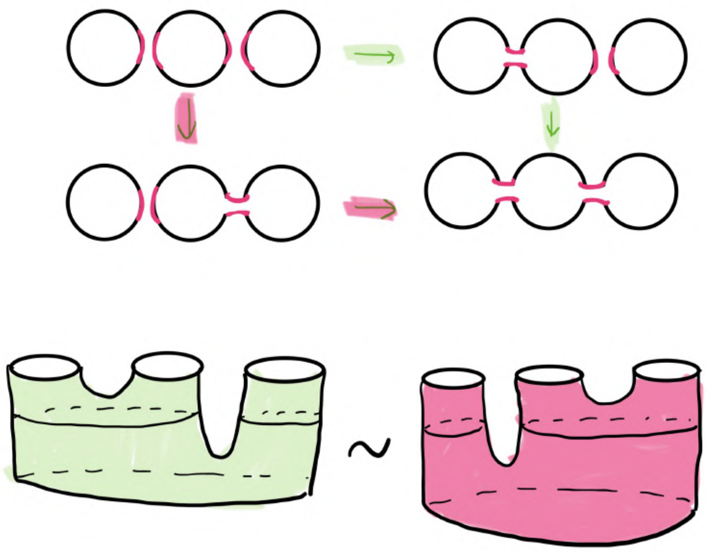


Figure 2.10

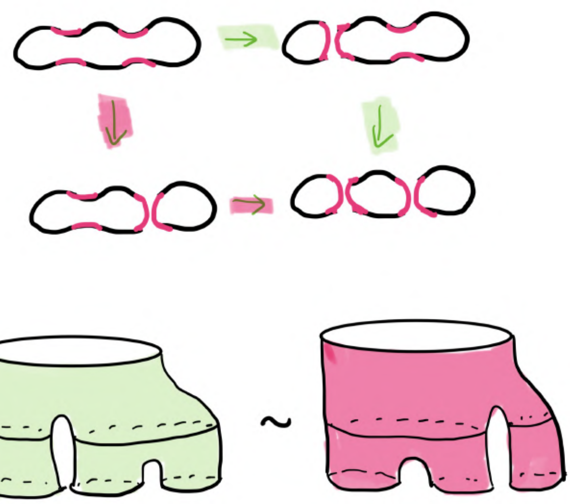


Figure 2.11

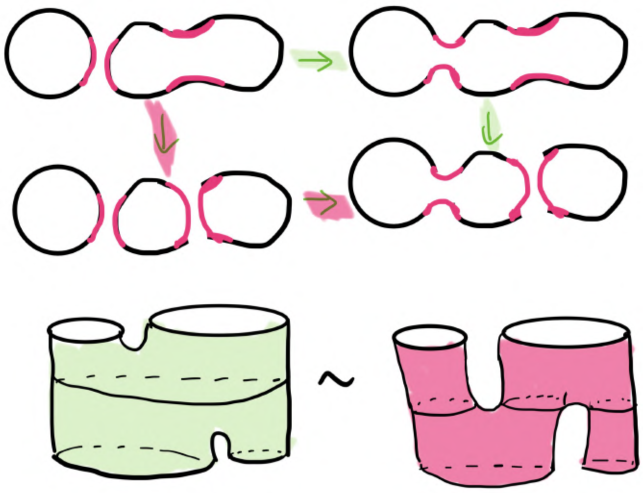


Figure 2.12

□

Applying a TQFT

We will apply a $(1+1)$ TQFT $\mathcal{A}$ to this cube: To each vertex associate the finitely-generated abelian group $\mathcal{A}(D_r)$ and to each edge e the linear map $\mathcal{A}(S_e)$. We do not explicit yet the TQFT $\mathcal{A}$ but think of it to be a TQFT over $\mathbb{Z}$ or $\mathbb{Q}$.
In particular since A is a functor, the 2-faces of this cube still commute!

By axioms of TQFT, to a collection of k disjoint circles one associates $A^{\otimes k}$. Also since the ground ring is $\mathbb{Z}$ we have $\mathcal{A}(\emptyset) = \mathbb{Z}$.

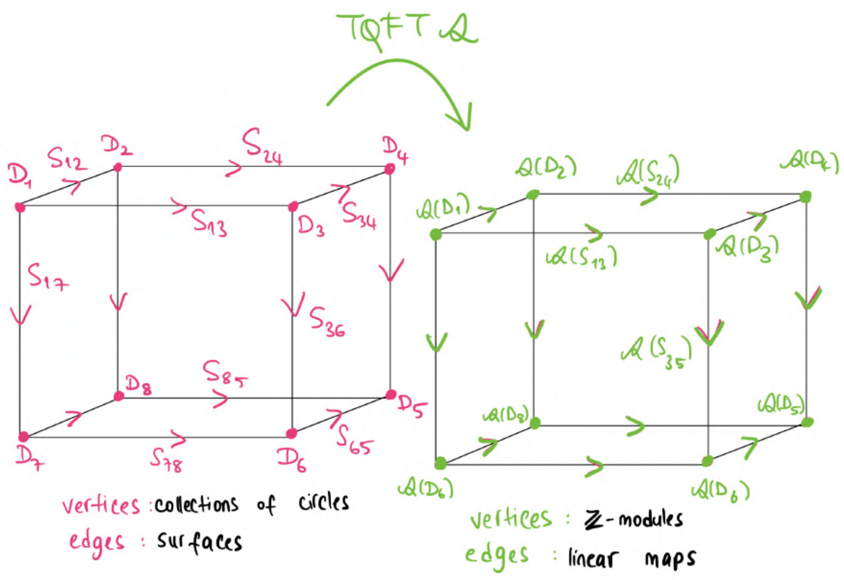


Figure 2.13

Definition 2.1. *Finally, define the Khovanov bracket of the (unoriented) diagram D as a group by*

$$\bar{C}(D) = \bigoplus_r \mathcal{A}(D_r)$$

where the sum runs over all the resolutions of D and the differential is given for $x \in \mathcal{A}(D_r)$ by

$$dx = \sum_{e:r \to r'} (-1)^{\sigma(e)} \mathcal{A}(S_e)(x)$$

where σ assigns to each edge $e : r \to r'$ the number of 1-resolutions on the left of the variable coordinate of the resolutions.

Lemma 2.2. *On each square of the cube, the number of edges with sign $-$ is odd.*

The TQFT $\mathcal{A}$ Let us give an explicit description of this TQFT $\mathcal{A}$ via the Frobenius system $(A, m, \Delta, \iota, \varepsilon)$:
Let $A = \mathbb{Z}.1 \oplus \mathbb{Z}.X$ be the free abelian group generated by two elements $\{1, X\}$.

In this TQFT the multiplication and comultiplication maps m, Δ associated respectively to cobordisms "merge" and "split" are given by:

$$m(1 \otimes a) = m(a \otimes 1) = a, \forall a \in A \qquad \Delta(1) = 1 \otimes X + X \otimes 1$$
$$m(X \otimes X) = 0 \qquad \Delta(X) = X \otimes X$$

The unit and counit maps ι, ε associated respectively to the cap and cup cobordisms are defined by:

$$\iota(1) = 1 \qquad \varepsilon(X) = 1$$
$$\varepsilon(1) = 0$$

By Morse theory any cobordism can be composed into handle attachments, so any morphism in Cob^3 can be built up as a composition of *elementary cobordisms* as below and their disjoint union with cylinders

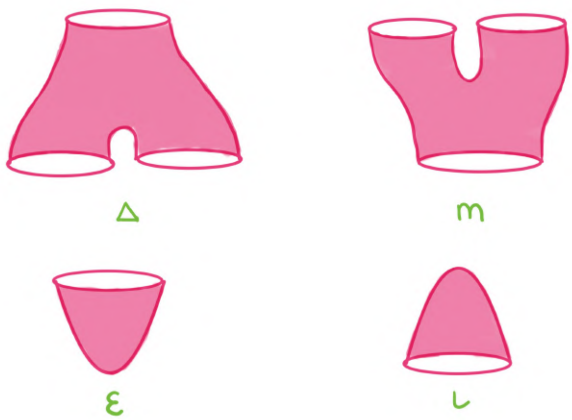


Figure 2.14

Hence $\mathcal{A}$ associates maps m, Δ, ι and ε to the elementary cobordisms and this determines completely the TQFT $\mathcal{A}$ on Cob^3. Below we check the well-definedness of $\mathcal{A}$

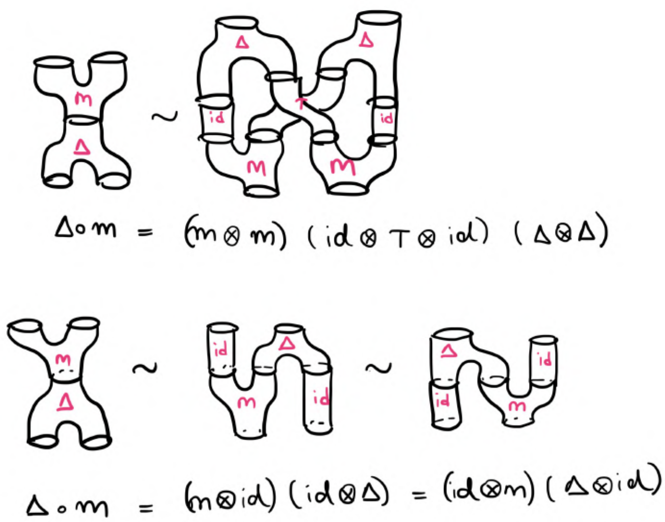


Figure 2.15

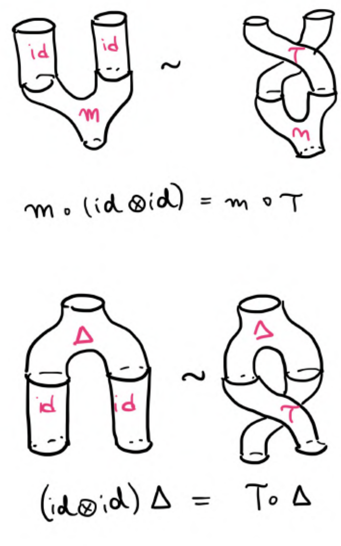


Figure 2.16

Example 2.1. *For example if e is an edge passing from a resolution r to r' where a circle splits into two circles, then $\mathcal{A}(S_e)$ is given by*

$$id^{\otimes(|D_r|-1)} \otimes \Delta : \mathcal{A}(D_r) \to \mathcal{A}(D_{r'})$$

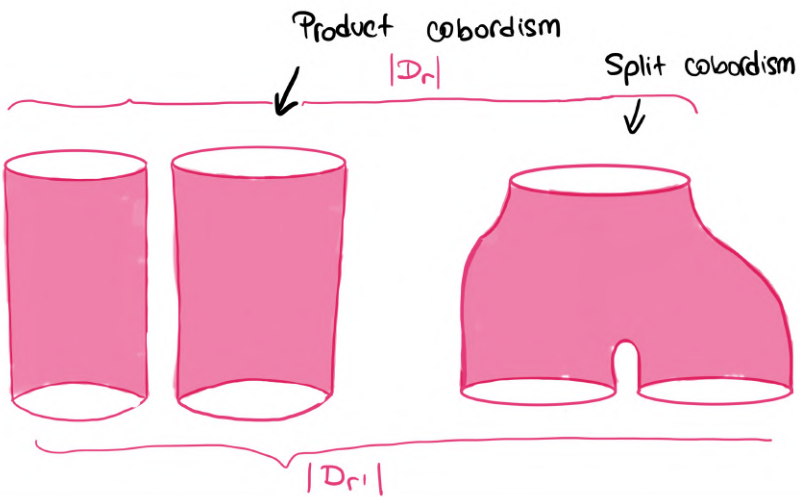


Figure 2.17

Gradings Let us make of A a graded $\mathbb{Z}$-module. We associate degrees to the generators by:

$$\deg(1) = -1, \deg(X) = 1$$

so that the graded rank qdim of A is $q + q^{-1}$.

We extend the degree to a homogeneous element in a k-fold tensor product of A by

$$\deg(x \otimes y) = \deg(x) + \deg(y)$$

Lemma 2.3. *With this grading, $\mathcal{A}$ is a graded TQFT in the sense that for any cobordism S,*

$$\deg(\mathcal{A}(S)(x)) = \chi(S) + \deg(x)$$

Proof. Indeed, as verified below, it holds for the 4 elementary cobordisms since any S is composed of them and that we have $\chi(S_1 \circ S_2) = \chi(S_1) + \chi(S_2)$

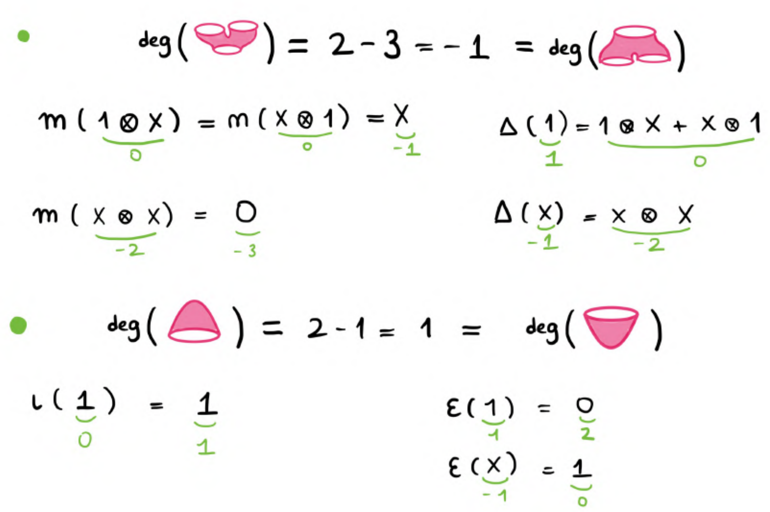


Figure 2.18

$\square$

Definition 2.2. *For $x \in \mathcal{A}(D_r)$ define*

$$q(x) = \deg(x) + |r|$$

This grading is called the quantum grading.

Remark 2.1. *Notice that the differential d preserves the quantum grading! Indeed $deg(d(x)) = deg(x) - 1$ and $|r'| - |r| = 1$.*

Now that the complex $\bar{C}(D)$ is graded, we can decompose it as a direct sum of chain complexes

$$\bar{C}(D) = \bigoplus_j \bar{C}^{*,j}(D)$$

Hence $\bar{C}(D)$ is a bigraded $\mathbb{Z}$-module. Throughout it will be useful to remember that in $\bar{C}(D)$, an element x in $\mathcal{A}(D_r)$ has for homological grading $gr_h(x) = |r|$ and quantum grading $q(x) = \deg(x) + |r|$.

The Khovanov complex To define a link invariant we let the orientations get back in the game. Indeed for Khovanov homology to be a link invariant we will need to consider shiftings by some quantities depending on the orientation.
Let D be an oriented link diagram of an oriented link L. Note $n_+(D), n_-(D)$ respectively the number of positive and negative crossings in the diagram.

Definition 2.3. *The Khovanov complex of the oriented diagram D is defined by*

$$CKh(D) = \bar{C}(D)[-n_-(D)]\{-2n_-(D) + n_+(D)\}$$

It is a bigraded complex of abelian groups. Its homology $H_*(CKh(D))$ is called Khovanov homology and we denote it as $Kh(D)$.

Remark 2.2. *To see that this complex is bigraded, recall that the map d preserves the quantum-grading. Then, shifting the quantum gradings of all space by a constant $n_+ - 2n_-$ does not effect the grading-preserving property. Hence the gradings descend to homology, and the Khovanov homology is a bigraded homology theory.*

Remark 2.3. *Remark that this shiftings are a mirroring of the formula of the unnormalized Jones polynomial via the cube of resolutions and the Kauffman bracket*

Still to keep track of the gradings, note that a homogeneous element x in $CKh^{i,j}(D)$ has homological degree $i = gr_h(x)$ and quantum degree $j = gr_q(x)$. If $x \in A(D_r) \cap CKh^{i,j}(D)$,

$$\begin{aligned} i &= |r| - n_- \\ j &= deg(x) + i + w(D) \end{aligned}$$

Remark 2.4. *Also, remark that the chain spaces are trivial for homological degree $i < -n_-$ and $i > n_+$. So, even though Khovanov homology doesn't give directly the number of crossings of a minimal diagram of a link L, it gives a lower bound on the minimal positive and negative crossings.*

Example 2.2. *It is time to see some examples on some basic knots so that we get a feeling for Khovanov homology [Lob24]:*

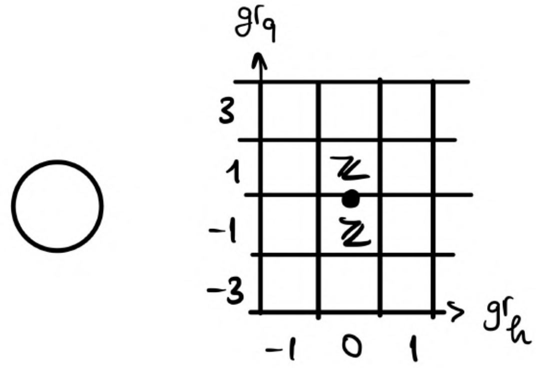


Figure 2.19

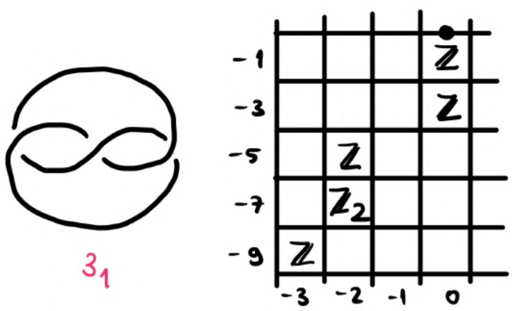


Figure 2.20

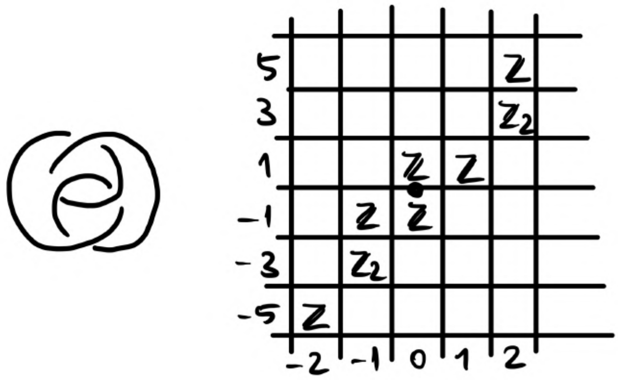


Figure 2.21

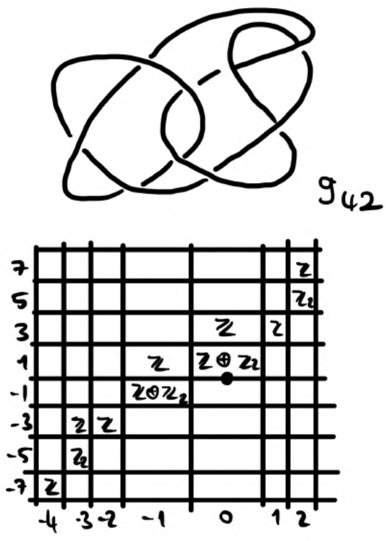


Figure 2.22

Theorem 2.1. *We have*

$$\chi(Kh(L)) = \bar{J}(L)$$

In other terms, the Khovanov homology categorifies the unnormalized Jones polynomial.

Proof. Indeed we just write the definition of the Euler characteristic of a bigraded complex. The graded Euler characteristic of this complex is

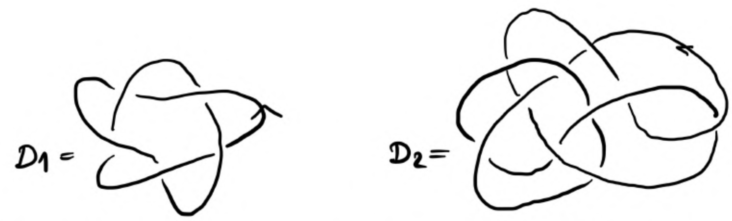
$$\chi(Kh(L)) = \sum_{i,j\in\mathbb{Z}} (-1)^i \mathrm{qdim}(Kh^{i,j}(D)) \in \mathbb{Z}[\pm 1]$$

Hence, by replacing i and j as in the previous remark, we have finally that:

$$\begin{aligned}
\chi(Kh(L)) &= \chi(CKh(D)) \\
&= \sum_{i,j\in\mathbb{Z}} (-1)^i \mathrm{qdim}(CKh^{i,j}(D)) \\
&= \sum_{r\in[0,1]^n} (-1)^{|r|-n_-}\ q^{|r|+n_+-2n_-}\mathrm{qdim}(\mathcal{A}(D_r)) \\
&= \sum_{r\in[0,1]^n} (-1)^{|r|-n_-}\ q^{|r|+n_+-2n_-}(q+q^{-1})^{|D_r|} \\
&= \bar{J}(L)
\end{aligned}$$

□

Khovanov homology is a stronger invariant then the Jones polynomial. Below are two knot diagrams D_1, D_2 having the same Jones polynomial but different Khovanov invariants. Hence, they represent different knots.



Khovanov homology $KH^{i,j}(D_1)$:

j \ i	-5	-4	-3	-2	-1	0
-3						$\mathbb{Q}$
-5						$\mathbb{Q}$
-7				$\mathbb{Q}$		
-9						
-11		$\mathbb{Q}$	$\mathbb{Q}$			
-13						
-15	$\mathbb{Q}$					

Khovanov homology $KH^{i,j}(D_2)$:

j \ i	-7	-6	-5	-4	-3	-2	-1	0
-1							$\mathbb{Q}$	$\mathbb{Q}$
-3								$\mathbb{Q}$
-5					$\mathbb{Q}$	$\mathbb{Q}\oplus\mathbb{Q}$		
-7				$\mathbb{Q}$				
-9				$\mathbb{Q}$	$\mathbb{Q}$			
-11		$\mathbb{Q}$	$\mathbb{Q}$					
-13								
-15	$\mathbb{Q}$							

Figure 2.23

Before doing an explicit computation of Khovanov homology, let us summarize how we obtain the Khovanov homology of a link L:

(1) Take a diagram D of a link and apply the $(1+1)$ TQFT to its cube of resolutions

(2) Flatten the cube of resolutions along the Hamming weights to obtain a graded complex $\bar{C}(D)$ of finitely-generated abelian groups.

(3) Shift the bigraded complex by $[-n_-(D)]\{n_+(D)-2n_-(D)\}$

Example 2.3. *Khovanov homology of the Hopf link*

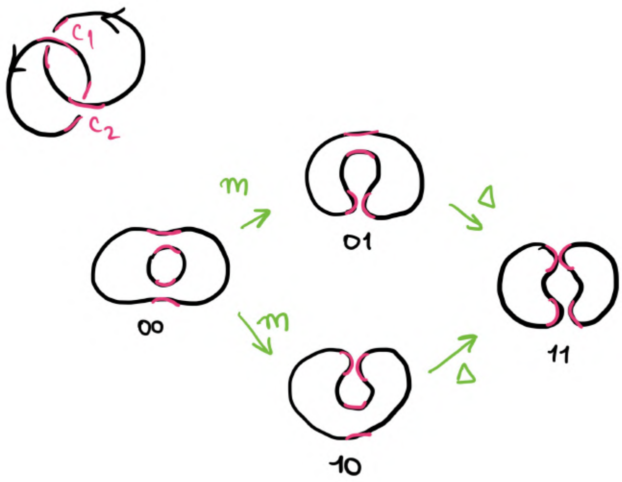


Figure 2.24

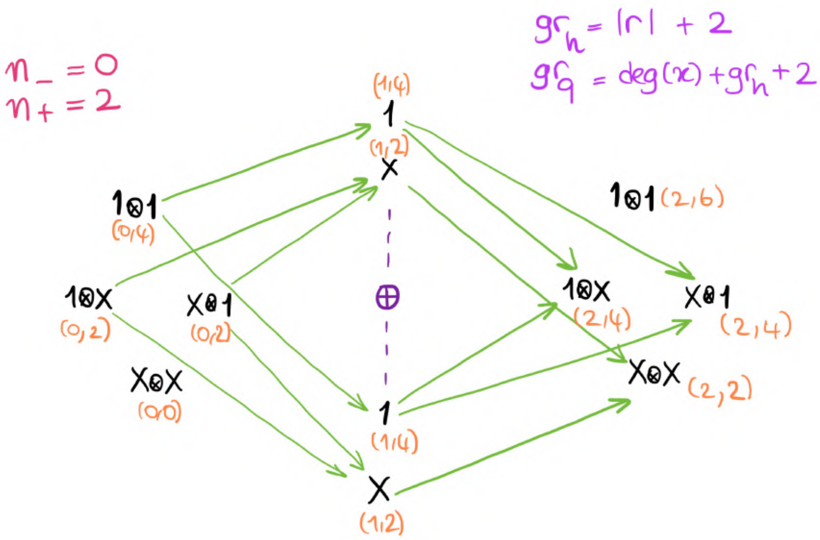


Figure 2.25

- $$\frac{\ker(d_0)}{\operatorname{Im}(d_{-1})} = \langle \underbrace{[X \otimes X]}_{(0,0)}, \underbrace{[1 \otimes X - X \otimes 1]}_{(0,2)} \rangle$$
- $$\frac{\ker(d_1)}{\operatorname{Im}(d_0)} = \langle 0 \rangle$$
- $$\frac{\ker(d_2)}{\operatorname{Im}(d_1)} = \frac{A^{\otimes 2}}{\langle X \otimes 1 + 1 \otimes X,\ X \otimes X \rangle} = \langle \underbrace{[1 \otimes 1]}_{(2,6)}, \underbrace{[X \otimes 1] - [1 \otimes X]}_{(2,4)} \rangle$$

Figure 2.26

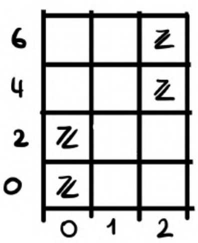


Figure 2.27

Some elementary properties of Khovanov homology

Here we mention some elementary properties of Khovanov homology that we will need later on.

Lemma 2.4. *Let L be a link with k components and let D be a link diagram with $n(D)$ crossings and $O(D)$ Seifert circles. Then*

$$O(D) - n(D) = k \pmod 2$$

Proof. Recall from 1.4 that from the Seifert algorithm one gets a handle decomposition of the Seifert surface S obtained by D having $O(D)$0-handles, $n(D)$1-handles. Moreover, the surface has Euler characteristic $\chi(S)$ so

$$O(D) - n(D) = 2 - 2g - k$$

so

$$O(D) - n(D) = k \pmod 2$$

In particular for a knot K, $O(D) - n(D)$ is always an odd integer. □

First of all, some properties seem to be common to all of the examples above. Observe that for the knots all quantum degrees are odd and for the Hopf link they are even. Indeed this is not a coincidence.

Proposition 2.1. *Let L be a link with k components. Then any quantum grading j vanishes if $j \neq k \pmod 2$.*

Proof. Let n_-, n_+ be respectively the number of negative and positive crossings in the diagram D. We first consider the Khovanov bracket and then apply the global shifting on quantum grading by $n_+ - 2n_-$.
Notice that in the cube of resolutions, moving from one vertex to other does not change the parity of the quantum grading. Indeed moving to another vertex via an edge the Hamming weight will be shifted by 1 and the number of circles in the resolved diagrams will be shifted by ± 1. Recall that in the Khovanov bracket the quantum grading is given by

$$gr_q(x) = deg(x) + gr_h(x)$$

where gr_h is given by the number of circles in the resolved diagram. Hence

$$\begin{aligned} gr_q(x) - gr_q(d(x)) &= deg(x) + gr_h(x) - deg(d(x)) - gr_h(d(x)) \pmod 2 \\ &= deg(x) + gr_h(x) - deg(x) - 1 - gr_h(x) \pm 1 \pmod 2 \\ &= 0 \pmod 2 \end{aligned}$$

Hence to simplify, let us take x to be on the vertex with Hamming degree (or equivalently homological grading) 0 . Suppose $x \in A(D_r)$ and there are $O(D_r)$ circles in that resolved diagram. Then

$$gr_q(x) = deg(x) = O(D_r) \pmod 2$$

Now passing to the Khovanov homology, shift the homological degree by $-n_-$ and quantum degree by $n_+ - 2n_-$ so we have

$$\begin{aligned} gr_q(x) &= O(D_r) + n_+ + n_- \pmod 2 \\ &= O(D_r) + n \pmod 2 \\ &= k \pmod 2 \end{aligned}$$

where the last equality is by the previous lemma. □

Proposition 2.2. *Let $\bar{D}$ be the mirror image of an oriented link diagram D. Then there is an isomorphism*

$$CKh(D)^* \cong CKh(\bar{D})$$

of graded chain complexes.

Proof. We show that we have an isomorphism between the cubes of resolved diagrams. Let $\{1^*, X^*\}$ be a basis for $A^* = Hom(A, \mathbb{Z})$ given by

$$\begin{aligned} &1^*(1) = 0, 1^*(X) = 1 \\ &X^*(1) = 1, X^*(X) = 0 \end{aligned}$$

Let $m*, \Delta^*$ maps defined by

$$\begin{aligned} &m^*(1^* \otimes X*) = m(X^* \otimes 1^*) = X^*, && \Delta^*(1^*) = 1^* \otimes X^* + X^* \otimes 1^* \\ &m^*(X^* \otimes X^*) = 0 && \Delta^*(X^*) = X^* \otimes X^* \end{aligned}$$

Define the isomorphism

$$\begin{aligned} \mu : A \to A^* \\ \mu(1) = 1^*, \mu(X) = X^* \end{aligned}$$

takes the maps m, Δ to m^*, Δ^* respectively. Hence $A \cong A^*$ as a Fröbenius algebra. Now we define r^* for r a resolution and $r*$ is the complementary resolution (obtained by $0 \mapsto 1, 1 \mapsto 0$). We claim that the number of resolved circles in D_r and $\bar{D}_{r^*}$ is the same hence by the $TQFT\mathcal{A}$. Moreover the maps m^*, Δ^* are dual respectively to maps Δ, m hence we get an isomorphism between $CKh(\bar{D})$ and $CKh(D)^*$. □

In particular, this gives us

Corollary 2.1. *Let L be an oriented link and $\bar{L}$ its mirror image. Over $\mathbb{Q}$*

$$Kh^{i,j}(\bar{L}) = Kh^{-i,-j}(L)$$

Proposition 2.3. *For a disjoint union $D \sqcup D'$ of two oriented link diagrams D and D'*

$$CKh(D \sqcup D') = CKh(D) \otimes CKh(D')$$

Proof. Any full resolution of $D \sqcup D'$ splits $r = r_1 \times r_2$ a product of a resolution of D and a resolution of D'. In particular, circles of a resolved diagrams split aswell and applying the TQFT, each module $A^{\otimes k}$ splits as $A^{\otimes k_1} \otimes A^{\otimes k_2}$.
Now let us check what happens at the level of shiftings and gradings. It is clear that we have

$$n_+(D \sqcup D') = n_+(D) + n_+(D'), n_-(D \sqcup D') = n_-(D) + n_-(D')$$

Now if an edge in the cube of resolution concerns two crossings of D, the sign is that of the differential of $CKh(D)$. Whenever the edge concerns the crossings of D', the sign of the differential in $CKh(D')$ must be multiplied by $(-1)^{|r_1|}$ where r_1 is the part of resolution living in the cube of D. All this discussion shows clearly that by definition of the tensor product of two chain complexes we have the result. $\square$

By the Künneth formula we get (over $\mathbb{Q}$):

Corollary 2.2. *For a disjoint union of two oriented links L, L',*

$$Kh^{m,k}(L \sqcup L') = \bigoplus_{i,j \in \mathbb{Z}} Kh^{i,j}(L) \otimes_{\mathbb{Q}} Kh^{k-i,m-j}(L')$$

In particular, notice that we can apply the same discussion on the Khovanov complexe of a disjoint union to the connected sum of links.

Remark 2.5. *Since under reversing orientation the sign of a crossing is unchanged, it is easy to see that*

$$Kh(L) = Kh(L^r)$$

for any link L.

Proposition 2.4. *Over $\mathbb{Q}$, for any two oriented knots K_1, K_2,*

$$Kh(K_1 \# K_2) \cong Kh(K_1 \# (K_2^r))$$

The most important property is that of invariance. Details of the proof can be found in [Kho00].

Theorem 2.2. *Let D and D' be two links diagrams related by a Reidemeister move. Then there exists a degree-0 isomorphism of bigraded $\mathbb{Z}$-modules*

$$\rho^* : Kh(D) \to Kh(D')$$

Proof.

- Invariance under the ordering of crossings: See [Kho00].
- Invariance under the three Reidemeister moves

Now let us prove the invariance under each Reidemeister move. We will define a map ρ_i at the chain level which will induce an isomorphism at the homology level.

Reidemeister 1 move Let D be a link diagram and D' the one with a twist. In particular, we have a new crossing. If we consider the cube of resolutions of D' and the Khovanov bracket defined by it, we can decompose the Khovanov bracket into a direct sum of sub-complexes as follows. Think of the added crossing c as the last coordinate in the cube of resolution of D'. Then the vertices of D' correspond to vertices of D followed by a resolution of c.

Denote $D'(*0), D'(*1)$ the two link diagrams obtained by D' by resolving only c by 0 and 1 resolution respectively.

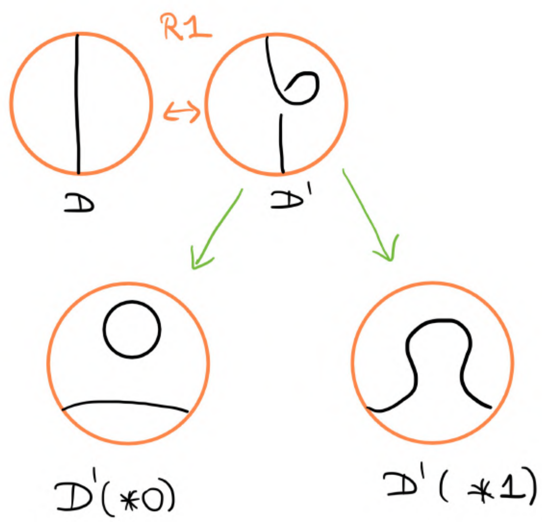


Figure 2.28: Invariance under $R1$ move

Now we can decompose $\overline{C}(D')$ as a group:

$$\overline{C}(D') = \overline{C}(D'(*0)) \oplus \overline{C}(D'(*1))[1]\{1\}$$

These shiftings come from the fact that in the cube of resolutions of D' one has one more 1 in each vertex where c is resolved by a 1-resolution, since in $D'(*1)$ we already consider that vertex as resolved.
Also denote $d'_{01} : \overline{C}(D'(*0)) \to \overline{C}(D'(*1))[1]\{1\}$ and d'_0, d'_1 similarly such that

$$d' = \begin{pmatrix} d'_0 & -d'_1 \\ d'_{01} & 0 \end{pmatrix}$$

So for $z \in \overline{C}(D'(*1))[1]\{1\}, y \in \overline{C}(D'(*0))$

$$d'(y+z) = d'_0(y) + d'_{01}(y) - d'_1(z)$$

From now on for the other Reidemeister moves we will adopt a similar notation.
Define

$$\begin{aligned} X_1 &= Ker(d'_{01}) \\ X_2 &= \{y \otimes 1 + z \mid y \in \overline{C}(D), z \in \overline{C}(D'(*1)[1]\{1\}\} \end{aligned}$$

Clearly as one can see in the above figure we have $\overline{C}(D(*0)) \cong \overline{C}(D) \otimes A$ and $\overline{C}(D') = X_1 \oplus X_2$ where X_2 is an acyclic complex. Indeed the differential restricted to X_2 is the map $y \otimes 1 \mapsto y$ so is an isomorphism. For $X_2 \cong Cone(d_{|X_2}$ and it is easy to see that the mapping cone of an isomorphism is acyclic.
Now define

$$\begin{aligned} \rho_1 : & X_1 \to \overline{C}(D)\{-1\} \\ & y \otimes 1 + z \otimes X \mapsto z \end{aligned}$$

which induces an isomorphism $Kh(D') \to Kh(D)$.

Reidemeister 2 move The idea for theReidemeister 2 move is similar, theonly difference is that we now need to take care of two additional crossings. Below are the associated local changes due to those crossings:

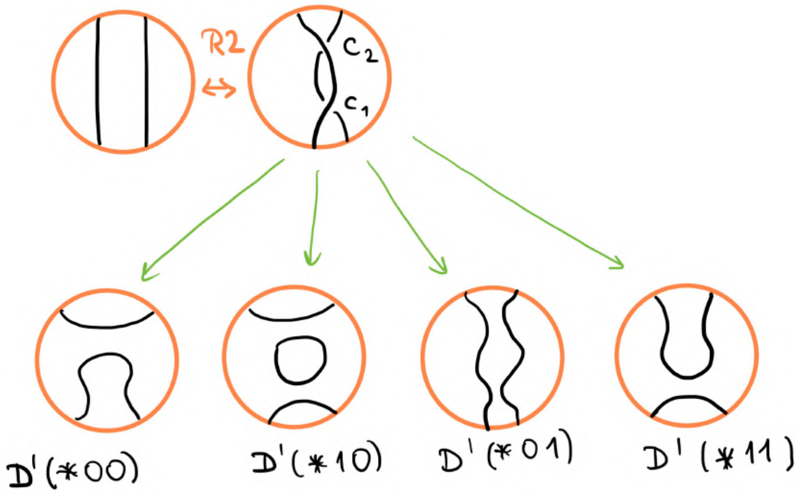


Figure 2.29

Define

$$\begin{aligned}
X_1 &= \{z + \alpha(z) \mid z \in \overline{C}(D'(*01)[1]\{1\}\} \\
X_2 &= \{z + d'_{00}y \mid z, y \in \overline{C}(D'(*00))\} \\
X_3 &= \{z + y \otimes 1 \mid z, y \in \overline{C}(D'(*11))[2]\{2\}\}
\end{aligned}$$

where $\alpha(z) = -d_{01\to 11}(z) \otimes 1$. Again, we have that X_2, X_3 are acyclic.

$$\overline{C}(D') = X_1 \oplus X_2 \oplus X_3$$

Then using the identification $\overline{C}(D)[1]\{1\} \cong \overline{C}(D'(*01))[1]\{1\}$ and the map ρ_2 where

$$\begin{aligned}
\rho_2 :& \overline{C}\hat{\imath}(D)[1]\{1\} \to X_1 \cap \overline{C}^i(D') \\
& z \mapsto (-1)^i(z + \alpha(z))
\end{aligned}$$

we have an isomorphism $Kh(D') \to Kh(D)$

Reidemeister 3 move: This time notice that the local move does not induce additional crossings, but instead they are *moving* and we have transformations at the level of resolved diagrams of D and D' as below

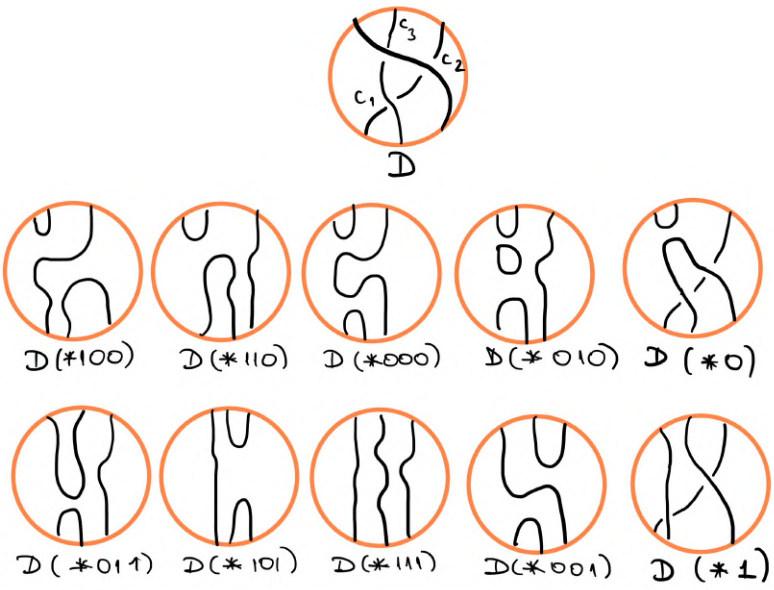


Figure 2.30

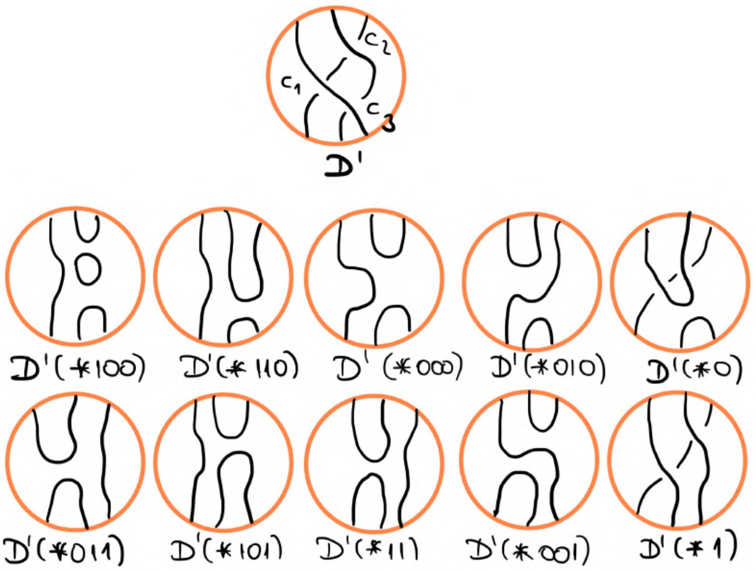


Figure 2.31

We define $\alpha, \beta, \gamma, \alpha', \beta', \gamma'$ as follows

$$\alpha : \overline{C}(D(*110))[2]\{2\} \to \overline{C}(D(*110))[1]\{1\} \otimes A, \alpha(z) = z \otimes 1$$
$$\beta : \overline{C}(D(*100))[1]\{1\} \to \overline{C}(D(*010))[1]\{1\}, \beta(z) = \alpha d_{100\to 110}(z)$$
$$\alpha' : \overline{C}(D'(*110))[2]\{2\} \to \overline{C}(D'(*100)[1]\{1\} \cong \overline{C}(D'(*110)[1]\{1\} \otimes A, \alpha'(z) = z \otimes 1$$
$$\beta' : \overline{C}(D'(*010))[1]\{1\} \to \overline{C}(D'(*100)[1]\{1\}, \beta'(z) = -\alpha' d'_{010\to 110}(z)$$

Moreover $\overline{C}(D), \overline{C}(D')$ can be decomposed as

$$\overline{C}(D) = X_1 \oplus X_2 \oplus X_3$$
$$\overline{C}(D') = X_1' \oplus X_2' \oplus X_3'$$

where these sub-complexes are given by

$$X_1 = \{x + \beta(x) + y \mid x \in \overline{C}(D(*100))[1]\{1\}, y \in \overline{C}(D(*1))[1]\{1\}\}$$
$$X_2 = \{x + d(y) \mid x, y \in \overline{C}(D(*000))\}$$
$$X_3 = \{\alpha(x) + d\alpha(y) \mid x, y \in \overline{C}(D(*110))[2]\{2\}\}$$
$$X_1' = \{x + \beta'(x) + y \mid x \in \overline{C}(D'(*010))[1]\{1\}, y \in \overline{C}(D'(*1))[1]\{1\}\}$$
$$X_2 = \{x + d'(y) \mid x, y \in \overline{C}(D'(*000))\}$$
$$X_3 = \{\alpha'(x) + d'\alpha'(y) \mid x, y \in \overline{C}(D'(*110))[2]\{2\}\}$$

Again, X_2, X_3, X_2', X_3' will be acyclic and we have isomorphisms

$$\overline{C}(D(*100))[1]\{1\} \cong \overline{C}(D'(*010))[1]\{1\}$$
$$\overline{C}(D(*1))[1]\{1\} \cong \overline{C}(D'(*1))[1]\{1\}$$

and $\rho : X_1 \to X_1$ an isomorphism defined by

$$\rho_3(x + \beta(x) + y = x + \beta'(x) + y$$

□

A mapping cone description

At this point, let us take a step back and give a mapping cone description for our Khovanov homology in terms of some mapping cone. This point of view will be useful to use tools from homological algebra.
Let D be a link diagram with at least 1 crossing. Now choose a crossing c.
Then we can split the cube of resolutions into two parts, decomposing vertices into a direct sum

$$F_0 \oplus F_1$$

where F_0 contains vertices having 0 on the first coordinate and F_1 those having 1. Now the edges of the cube lie in three disjoint sets:

$$E_0, E_{01}, E_1$$

where E_i contains edges between two vertices in F_i and E_{01} those from a vertex in F_0 to one of F_1.
Notice that a generator of $CKh(D)$ corresponds to a vertex r and an element in $\mathcal{A}(D_r)$. So this decomposition of the cube yields to a decomposition of the Khovanov bracket

$$\bar{C}(D) = C_0(D) \oplus C_1(D)$$

where $C_0(D)$ has generators living in F_0 and C_1 in F_1. Then the differential d in $CKh(D)$ decomposes

$$\begin{pmatrix} \partial_0 & 0 \\ \partial & \partial_1 \end{pmatrix}$$

where ∂_i are restrictions to edges in E_i *i.e.* $\partial_i = \mathcal{A}(e), e \in E_i$ and ∂ to E_{01}.

In order to see how one can identify $CKh(D)$ with a mapping cone, let us recall our definition of a mapping cone of a chain map.
If $d : C^1 \to C^2$ is a chain map between two chain complexes then $Cone(d)$ is defined by

$$Cone(d)_n = C_1^n \oplus C_0^{n+1}$$

and yields to a short exact sequence

$$0 \to C_1 \to Cone(d) \to C_0[-1] \to 0$$

Now let D_0 and D_1 be diagrams of D where the crossing has resolved by 0 or 1 resolution respectively. Notice that we can identify

$$C_0 \cong \bar{C}(D_0)$$
$$C_1 \cong \bar{C}(D_1)[1]\{1\}$$

. These shiftings on $CKh(D_1)$ come from the fact that in the cube of D, a vertex living in F_0 has one more 1-coordinate then that living in D_1. In particular the chain maps are quantum degree-preserving. Now we have $CKh(D) \cong Cone\left(\partial : CKh(D_0) \to CKh(D_1)[1]\{1\}\right)\{k\}$ where k is an overall shifting depending on the sign of the chosen crossing. So we have a short exact sequence

$$0 \to CKh(D_1)[1]\{k+1\} \to CKh(D) \to CKh(D_0)[-1]\{k\} \to 0$$

giving a long exact sequence

$$\cdots \to Kh^{i-1}(D_1)\{k+1\} \to Kh^i(D) \to Kh^{i+1}(D_0)\{k\} \xrightarrow{d_{0\to 1}} Kh^{i+1}(D_1)\{k+1\} \to \cdots$$

Corollary 2.3. *Let K_1, K_2 be oriented knots. We have a short exact sequence*

$$0 \to CKh(K_1\#K_2^r)[1]\{2\} \xrightarrow{i} \mathrm{CKh}(K_1\#K_2) \xrightarrow{p} CKh(K_1 \sqcup K_2)[-1]\{1\} \to 0$$

In particular we have a long exact where chain maps are grading preserving sequence

$$\cdots \to Kh(K_1\#K_2)\{2\} \xrightarrow{i*} Kh(K_1\#K_2) \xrightarrow{p*} Kh(K_1 \sqcup K_2)\{1\} \xrightarrow{d} Kh(K_1\#K_2)\{2\} \to \cdots$$

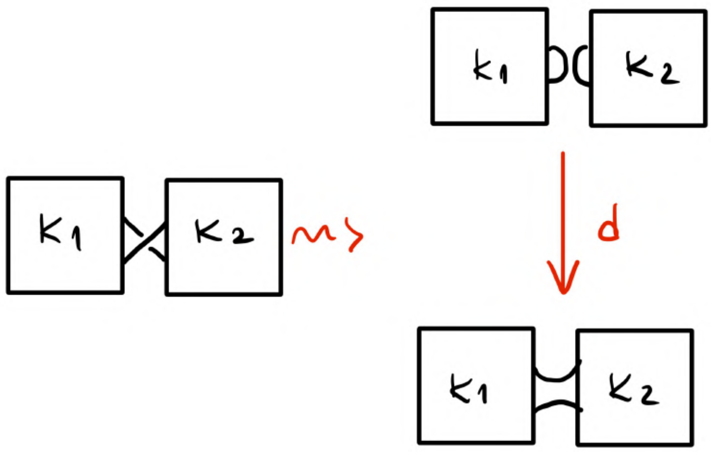


Figure 2.32

Proof. If we choose the central crossing in the diagram of $K_1\#K_2$ then this yields to a short exact sequence

$$0 \to CKh(K_1\#K_2^r)\{2\} \xrightarrow{i} \mathrm{CKh}(K_1\#K_2) \xrightarrow{p} CKh(K_1 \sqcup K_2)\{1\} \to 0$$

giving a long exact sequence

$$\cdots \to Kh^i(K_1\#K_2^r)\{2\} \to Kh^i(K_1\#K_2) \to Kh^{i+1}(K_1 \sqcup K_2)\{1\} \to Kh^{i+1}(K_1\#K_2^r)\{2\} \to \cdots$$

$\square$

Link cobordisms and maps of homology groups

Recall that instead of using links themselves, one projects on the plane and study link diagrams. But a representation given by a diagram is not unique, and two diagrams represent isotopic links if they are related by a finite number of Reidemeister moves. Now, link cobordisms between two given links can have a similar relation.

Definition 2.4. *A link cobordism* (Σ, L_0, L_1) *from* L_0 *to* L_1 *is a smooth, compact, oriented surface* S *properly embedded in* $\mathbb{R}^3 \times I$ *such that* $\partial\Sigma = \bar{L_0} \sqcup L_1$ *and* $\partial\Sigma \subset \mathbb{R}^3 \times \{0, 1\}$

A link cobordism from L_0 to L_1 can be represented by a finite sequence of oriented link diagrams: the first diagram being that of L_0 and the last one being for L_1. Two consecutive diagrams in this sequence must be related by a set of moves which is:

- Reidemeister I,II,III-moves
- Morse 0-,1-, 2- handle moves

Reidemeister moves are the usual ones. Below are the Morse moves:

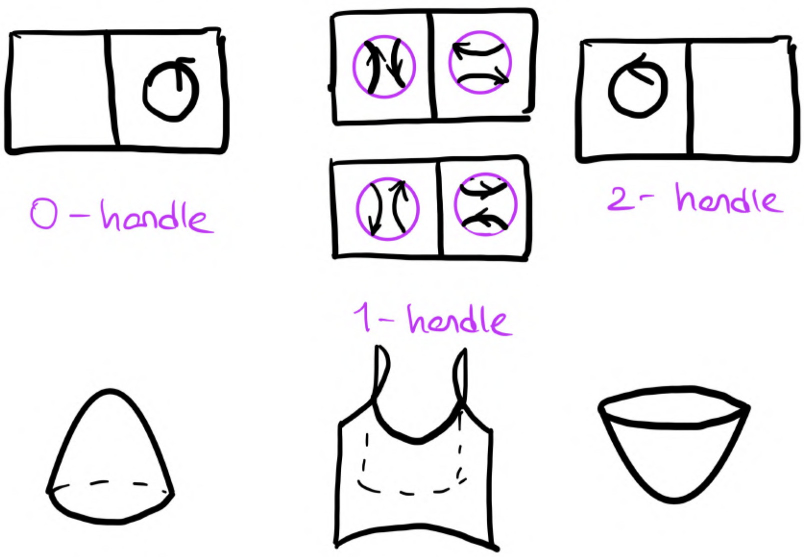


Figure 2.33: Morse moves

Geometrically, so in terms of cobordisms, Morse moves correspond to a cap, cup or saddle, so to attaching a 0-handle, 1-handle or 2-handle to the identity cobordism.

We call such a sequence a *movie*. Below is an example of a movie between the Hopf link and the empty link :

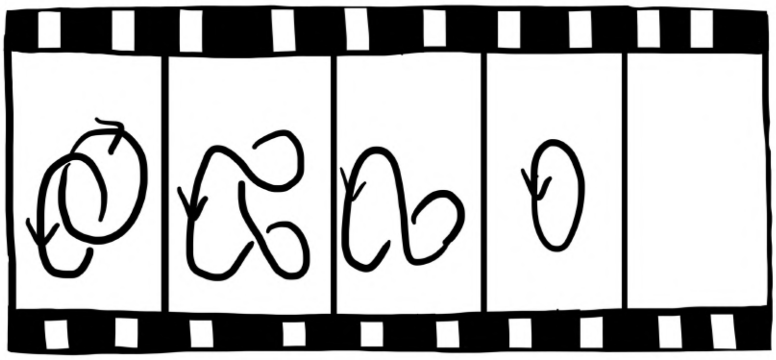


Figure 2.34: A movie between the Hopf link and the empty link

Proposition 2.5. *Let S be a cobordism between links L_0, L_1 and D_0, D_1 respective link diagrams. A movie (M, D_0, D_1) induces a map on Khovanov homology (over $\mathbb{Q}$)*

$$\phi_M : Kh^{*,*}(D_0) \to Kh^{*,*+\chi(S)}(D_1)$$

Proof. The idea of the proof is to define a map between the homology groups of each consecutive frames of the movie and then compose them to get the whole map.

Reidemeister moves: We take the maps ρ_i given by the proposition 2.6.

Morse moves: We will define maps at the chain level. To do this, consider the two cubes of resolution corresponding to D and D'. We define a cobordism between two diagrams which lie on the same vertex of the disjoint cubes as illustrated below, and then we apply the TQFT $\mathcal{A}'$. For each vertex, the map will be of the same graded degree and in the end we will get a graded map at the chain level.

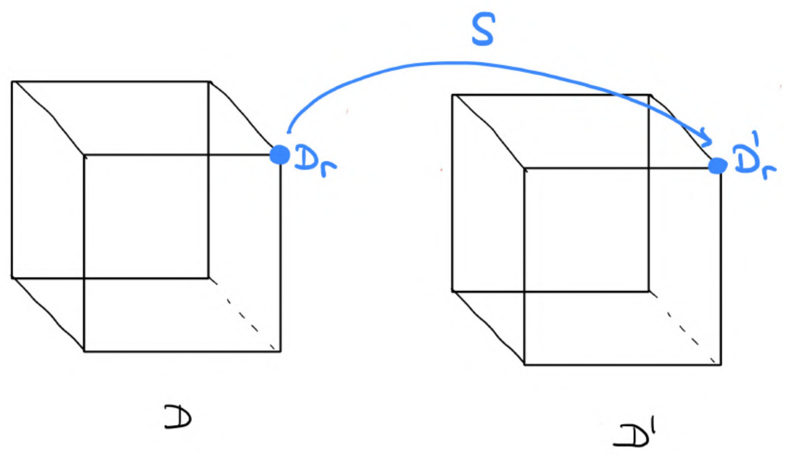


Figure 2.35

- **0-handle move(the birth move):** This move consists of adding a circle to a diagram D. The corresponding edge is a cup cobordism and the TQFT associates to it the unit map $\iota : R \to A$ of the algebra A:

$$id \otimes \iota : CKh(D) \to CKh(D')$$

of degree $\chi(S_e) = 1$.

- **1-handle move:** We construct a map $CKh^{*,*}(D) \to CKh(D')^{*,*-1}$ using the TQFT $\mathcal{A}$
Consider a resolution r of D. This induces a resolution of D' since the two diagrams do not differ at crossings. Take the cobordism S between D_r and D'_r composed of a merge or split cobordism in the neighbourhood where the Morse move takes place and of product cobordisms out of that neighbourhood. Considering the cube of resolutions, so applying this for each resolution and then applying the $TQFT\mathcal{A}$, we get a map of bigraded degree $(0, -1)$ at the chain level.

$$A(\mathcal{C}) : C^{i,j}(D) \to C^{i,j-1}(D')$$

Then one gets induced maps of degree $\chi(S_e)$ between homologies.

$$Kh^{*,*} \to Kh^{*,*-1}(D')$$

- **2-handle move (the death move):** This move consists of removing a circle from a diagram D, hence the edge is the cap cobordism of degree +1. Our TQFT associates to it the counit map of the algebra A which is of degree $\chi(S_e) = 1$

$$id \otimes \varepsilon : CKh(D) \otimes_{\mathbb{Q}} A \to CKh(D) \otimes_{\mathbb{Q}} \mathbb{Q}$$

and hence between homologies.

Now composing all these maps, we get a map $\phi_M : Kh^{*,*}(D_0) \to Kh^{*,*+\chi(M)}(D_1)$. From what follows this map is of degree $\chi(M)$ since Reidemeister maps are degree preserving and we have that $\chi(M) = \#0 - handles - \#1handles + \#2 - handles$.

□

Here is a little application of the discussion above. We can define a numerical invariant of closed oriented surfaces embedded in $\mathbb{R}^4$. Indeed, such a surface Σ can be viewed as a link cobordism between the empty link and the empty link. Since $Kh^{*,*}(\emptyset) = \mathbb{Q}$, we get a map $\phi_\Sigma : \mathbb{Q} \to \mathbb{Q}$ and $KJ_\Sigma = |\phi_M(1)|$ is the *Khovanov-Jacobsson number* .
Nevertheless, these numbers are pretty disappointing. If $\chi(\Sigma) \neq 0$, then $KJ_\Sigma = 0$ since ϕ shifts the q-degree by $\chi(\Sigma)$ and the only non zero terms are in bidegree $(0, 0)$ For embedded tori we have

Proposition 2.6. *If Σ is a smoothly embedded torus in $\mathbb{R}^4$, then $KJ_\Sigma = 2$*

.

Functoriality of Kh It is now natural to ask if, given two movie respresenttaions of a same cobordism, the maps ϕ_1, ϕ_2 coincide. In [CMW07] is given a modified version of Khovanov homology so that it becomes a functor. Alternatively in [San08] is given modifications on the signs of the maps without changing the setting. In any case here, we only need it to be an up-tp-sign functor.

2.2 Lee-Khovanov homology

2.2.1 Lee's homology theory

In his paper [Bar02] Bar-Natan gives a phenomenological conjecture:

Conjecture 2.1 ([Bar02])**.** *For an alternating link K, there exists $s(K) \in \mathbb{Z}$ such that*

$$P_{Kh(K)}(t, q) = q^{s(K)}(q + q^{-1}) + (1 + tq^4)Q(K)$$

where $Q(K)$ is a Laurent polynomial with only positive coefficients

This conjecture motivated Lee to define what we call Lee-Khovanov homology. In [Lee02] she proved the conjecture for alternating knots:

Theorem 2.3 (Lee). *For an alternating knot L, its Khovanov invariants $Kh^{i,j}(L)$ of degree difference $(1,4)$ are paired except in the 0th homology group. That is:*

$$Kh(L)(t,q) = q^s(q+q^{-1}) + (1+tq^4).C(t,q)$$

for some $s \in \mathbb{Z}$ and some polynomial C.

To prove it, she defined an endomorphism ϕ of bigraded degree $(1,4)$ from the Khovanov homology $Kh(L)$ of an oriented link to itself, which pairs most of $Kh(L)$. One can add ϕ to the Khovanov differential and obtain a new differential, hence a new homology called Lee-Khovanov homology which will be denoted by $Lee(L)$ throughout these notes.

Definition 2.5. *At the chain level, ϕ is defined in terms of multiplication and comultiplication maps by*

$$m_\phi(1 \otimes a) = 0 \ \forall a \in A \qquad \Delta_\phi(1) = 0$$
$$m_\phi(X \otimes X) = 1 \qquad \Delta_\phi(X) = 1 \otimes 1$$

To show that ϕ is defined on $Kh(L)$, we check the anticommutativity with d and the invariance under the Reidemeister moves, so that it commutes with the isomorphisms ρ_i. Details can be found in [Lee02].
Now, observe that

$$(\phi + d)^2 = \phi^2 + \phi d + d\phi + d^2 = 0$$

Also, the map $\phi + d$ behaves simply with a certain basis which we introduce now. Let

$$a = X + 1 \qquad b = X - 1$$

We have:

$$m_{\phi+d}(a \otimes a) = 2a \qquad \Delta_{\phi+d}(a) = a \otimes a$$
$$m_{\phi+d}(a \otimes b) = m_{\phi+d}(b \otimes a) = 0 \qquad \Delta_{\phi+d}(b) = b \otimes b$$
$$m_{\phi+d}(b \otimes b) = -2b$$

Definition 2.6. *Let D be an oriented link diagram. Lee-Khovanov chain complex of D, denoted $CLee(D)$ is the chain complex $(CKh(D), \phi + d)$ over $\mathbb{Q}$. We call its homology the Lee-Khovanov homology and denote $Lee(D)$.*

From now on we denote $d' := d + \phi$. Also denote m', Δ' respectively the maps $m_{\phi+d}, \Delta_{\phi+d}$ Let us explore the homology theory of d' in terms of TQFT and Frobenius algebra. Observe:

$$\begin{aligned} &m'(1 \otimes 1) = 1 && \Delta'(1) = X \otimes 1 + 1 \otimes X \\ &m'(1 \otimes X) = m'(X \otimes 1) = X && \Delta'(X) = X \otimes X + 1 \otimes 1 \\ &m'(X \otimes X) = 1 && \end{aligned}$$

Also take the unit and counit same as in Khovanov's TQFT:

$$\begin{aligned} &\iota'(1) = 1 && \varepsilon(1) = 0 \\ & && \varepsilon(X) = 1 \end{aligned}$$

Remark 2.6. *One can show that we have a Frobenius system (A', m', ι') and A' is the same as A as a $\mathbb{Q}$-vector space. But as an algebra, it is isomorphic to $\mathbb{Q}[X]/(X^2 - 1)$.*

Now observe that d' is not grading-preserving, but observe that it maps a homogeneous element x to a linear combination of homogeneous elements which have their degree equal or greater than that of x . This is useful to us because using it, one can still define a grading which we call the *filtration grading* on $CLee(D)$ as follows:

Definition 2.7. *Let $x \in CLee(D)$ and write $x = \sum_i^k a_i x_i$ as a linear combination of homogeneous elements with $a_i \in \mathbb{Q}, x_i \in CLee(D)$. Then $gr_q(x_i)$ makes sense and define the filtration grading*

$$gr_f(x) := \min(gr_q(x_i))$$

This grading induces a filtration by:

$$F^i(CLee(D)) := \{x \in CLee(D) | gr_f(x) \geq i\}$$

In particular for this filtration, m' and Δ' are filtered of degree -1 and ε, ι are filtered of degree -1. This is to say that that for the filtration grading, m' and Δ' are grading-preserving maps. Notice that for a homogeneous element x one has $gr_f(x) = gr_q(x)$, and we will sometimes abuse of notation and refer to the filtration grading of a non-homogeneous element as its quantum grading as an extension.

Observe that in particular we have

$$F^i(CLee(D)) = 0 \ \forall i > \max(gr_q(x)|x \in CKh(D))$$
$$F^i(CLee(D)) = CLee(D) \ \forall i < \min(gr_q(x)|x \in CKh(D))$$

In terms of the filtration, the filtration grading corresponds to

$$gr_f(x) = \max(i \in \mathbb{Z}|x \in F^i(CLee(D))$$

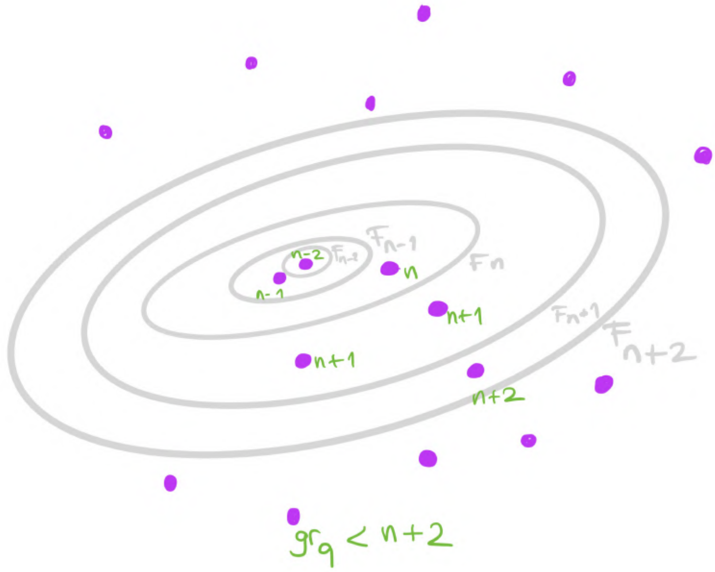


Figure 2.36

Remark 2.7. *Notice that $gr_f(x)$ corresponds to the integer i such that $x \in F^i(CLee(D))$ but $x \notin F^{i+1}(CLee(D))$.*

Moreover, this filtration induces a filtration grading and hence a filtration on $Lee(D)$:

$$gr_f([x]) = \max(gr_f(y) \mid [x] = [y])$$

$$F^i(Lee(L)) = \{[x] \in Lee(L) \ , \ gr_f([x]) \geq i\}$$

Observe that $[x] \in F^i(Lee(L))$ if and only if there exists $y \in CLee(D)$ such that $[x] = [y]$ and $y \in F^i(CLee(D))$.

Remark 2.8. *$gr_f([x])$ corresponds to the integer such that $[x]$ has a representative in $F^i(CLee(D))$ but not in $F^{i+1}(CLee(D))$.*

Theorem 2.4 (Invariance of Lee homology)**.** *Let D and D' be two link diagrams which differ by a Reidemeister move. Then we have an isomorphism of filtered vector spaces*

$$Lee(D) \cong Lee(D')$$

The proof is by defining, in the same way as we did in invariance of Khovanov homology, to define filtered maps ρ_i' for each Reidemeister move using the maps m', Δ' which induce isomorphisms at the homology level. See [Lee02] for details.

Now let us see some important properties of Lee-Khovanov homology:

Canonical generators

In [Lee02], Lee establishes a bijection with the 2^{n-1} orientations of a link L and a set of generators of $Lee(L)$, and she calls those generators the *canonical generators* since they are induced canonically by the orientations as follows.
Take a link diagram D and consider a checkboard colouring of the link diagram. Take an orientation $\mathfrak{o}$ of L and apply the oriented resolution. Now we have a collection of coloured disks with oriented boundary circles. Order the circles. Define

$$a = 1 + X$$
$$b = X - 1$$

Label a disk by a if when placing a point on its left (depending on its orientation) the point lives in a coloured zone and label it b otherwise. We associate a state $s_{\mathfrak{o}} \in CLee(L)$ to it by taking the tensor products of a's and b's. Below is an example

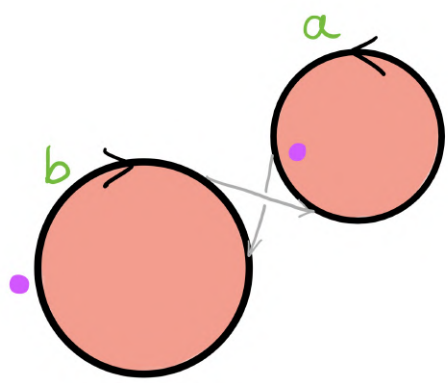


Figure 2.37: Canonical generators

Lemma 2.5. *Let D be an oriented link diagram. If in a disk neighbourhood of D there are exactly two segments, only one of the following can happen:*

- *They have parallel orientations and different labelings*
- *They have opposite orientations and same labelings*

Proof. Locally there are three cases:

(1) They belong to two circles not nested into each other. Attention, in this case the disks are both coloured or both uncoloured:

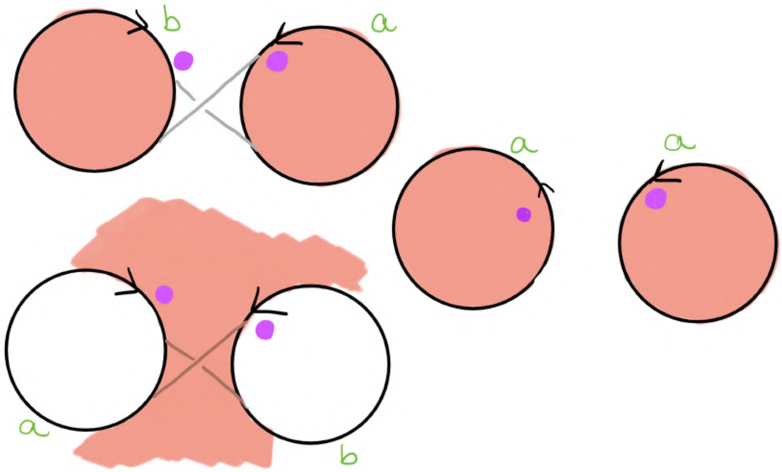


Figure 2.38

(2) They belong to circles nested one into another. In this case the checkboard colouring of the two circles will be opposite

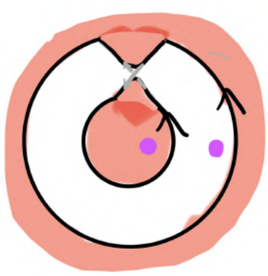


Figure 2.39

(3) They belong to the same circle, hence they have same labelling.

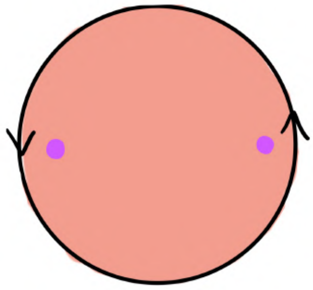


Figure 2.40

□

Proposition 2.7. *A state $s_{\mathfrak{o}}$ is closed: $d'(s_{\mathfrak{o}}) = 0$.*

Proof. Recall that $a = 1 + X$ and $b = 1 - X$ and $A' \cong \mathbb{Q}[X]/(X^2 - 1)$ as an algebra.
Let D be a link diagram of L with $n \in \mathbb{N}$ crossings and $\mathfrak{o}$ be an orientation of D. In the diagram induced by the oriented resolution suppose there are k circles. Then the state $s_{\mathfrak{o}}$ is in the k-fold tensor product $A' \otimes \cdots \otimes A'$.
Suppose that there is no crossing, then the differential is trivial and we are done.
Suppose that there is at least one crossing. We will show that the differential is defined by the multiplication map. Indeed, observe that in the oriented resolution the only crossings which are resolved by using a 0-resolution are the positive crossings. Hence, when we change a 0-resolution to a 1-resolution, two circles merge into one circle as below

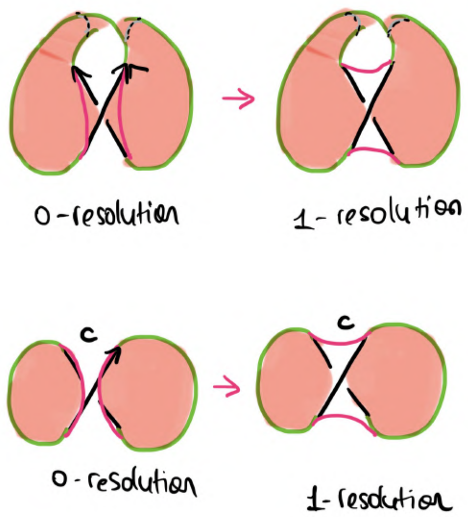


Figure 2.41

Hence the only map which appears in the differentials are multiplication map m and Id. Moreover, in oriented resolution any two strands in a crossing neighbourhood have the same orientation, hence they should have different labels. Finally we use $m(a \otimes b) = m(b \otimes a) = 0$ $\square$

Notice that, to define the Lee homology of a link L we have to fix an orientation on a diagram D, but the canonical generators are obtained by considering all the orientations. Nevertheless, the state corresponding to the fixed orientation of D has fixed homological degree.

Lemma 2.6. *Let D and D' be two knot diagrams which differ by the ith Reidemeister move. If o is an orientation on D, it induces an orientation o' on D'. Then ρ_i^* sends the canonical generator $[\mathfrak{s}_o]$ to a nonzero multiple of $[\mathfrak{s}_{o'}]$.*

Proof. Indeed since the maps ρ_i' are filtered maps and that they induce isomorphisms at homology level. Since $dim(Lee(K)) = 2$ for a knot and since we know that $s_{min}(K) \neq s_{max}(K)$, each sub-vector space in the filtration decomposition is 1-dimensional. By the filtration preserving argument we should have $\rho_i(\mathfrak{s}_o) \in \mathbb{Q}[\mathfrak{s}_{o'}]$. $\square$

Lemma 2.7. *Let $\mathfrak{o}$ be the fixed orientation on the link diagram D. Then the state $s_{\mathfrak{o}}$ lives in homology degree* 0*.*

In particular for a knot, Lee homology is only supported at homological level 0. This fact is simple yet crucial. For a knot, all the information is encoded in homological degree 0.

Proof. Indeed, note n the number of crossings in D, n_+ and n_- the number of positive and negative crossings respectively. Now for a positive crossing, the oriented resolution correspond to a 0-resolution whereas for a negative crossing the oriented resolution corresponds to the 1-resolution. In the complex $\bar{C}(D)$ this state lives in homological degree $n_-(D)$. But since we shift it by $-n_-(D)$ to obtain the homology, in $Lee(L)$ the state lives at homological degree 0. □

Remark 2.9. *Also, notice that if $\mathfrak{o}$ is an orientation on D and if $\bar{\mathfrak{o}}$ denotes the opposite orientation then the corresponding states live on the same resolved diagram. Indeed the checkboarder colouring is fixed, and since when one reverses the orientation at all the crossings at the level of resolved diagram this is reversing the orientations of all the boundary circles, so it is performing the transformations $a \leftrightarrow b$.*

The Rasmussen invariant will be defined using this theorem and the Khovanov-Lee spectral sequence which we define below.

Lemma 2.8. *Let L be a n-component link. Then each distinct orientation of L gives rise to two linearly independant generators of $Lee(L)$. Since there is 2^n many, in particular*

$$2^n \leq dim(Lee(L))$$

Moreover using canonical generators one can show that we have exactly 2^n generators, not more.

Lemma 2.9. *Choose a crossing of a link diagram D and consider the link diagrams $D(*0)$ and $D(*1)$ where the crossing has been resolved by $0-, 1-$ resolution. Then there is a short exact sequence*

$$0 \to CKh(D(*1))[1]\{1\} \to CKh(D) \to CKh(D(*0)) \to 0$$

Theorem 2.5 (Lee's structure theorem)**.** *Let L be a n-component link. Then*

$$\dim(Lee(L)) = 2^n$$

Proof. The proof is by strong induction on the number of components n and on the number of crossings c.

Suppose that L is a knot, or a 2-component link. If $c = 0$ then K is the unknot and we know that

$$Lee(K) = \mathbb{Q} \oplus \mathbb{Q}$$

hence $\dim(Lee(K)) = 2$.
Now suppose that L is a knot with c crossings. Choose a crossing c so that when we

resolve it one ow two obtained diagrams is a knot and the other is a 2-component link. We know by 2.9 we have a long exact sequence

$$\cdots Lee^k(D(*0)) \xrightarrow{d_{0\to 1}} Lee^k(D(*1))[1]\{1\} \to Lee^{k+1}(D) \to Lee^{k+1}(D(*0)) \to \cdots$$

Then since we have

$$\begin{aligned} dim(Lee(D)) &= dim(Lee(D(*0)) + dim(Lee(D(*1)) - 2\ rank(d_{0\to 1}) \\ &\leq dim(Lee(D(*0)) + dim(Lee(D(*1)) \end{aligned}$$

Suppose that $D(*1)$ is a knot and $D(*0)$ is a link. Orient the knot $D(*1)$ however you like, say with o. On $D(*0)$ there is an orientation $\tilde{o}$ which is compatible with that of $D(*0)$. This means that when we change the resolution of the distinguished crossing, we obtain $D(*1)$ with the chosen orientation. Similarly there is also an orientation on $D(*0)$ which is compatible with that of D. Observe that under the induced map by $d_{0\to 1}$, the generator $\mathfrak{s}_{\tilde{o}}$ is mapped to a non-zero multiple of $\mathfrak{s}_o$. Below we give an example on the right handed trefoil

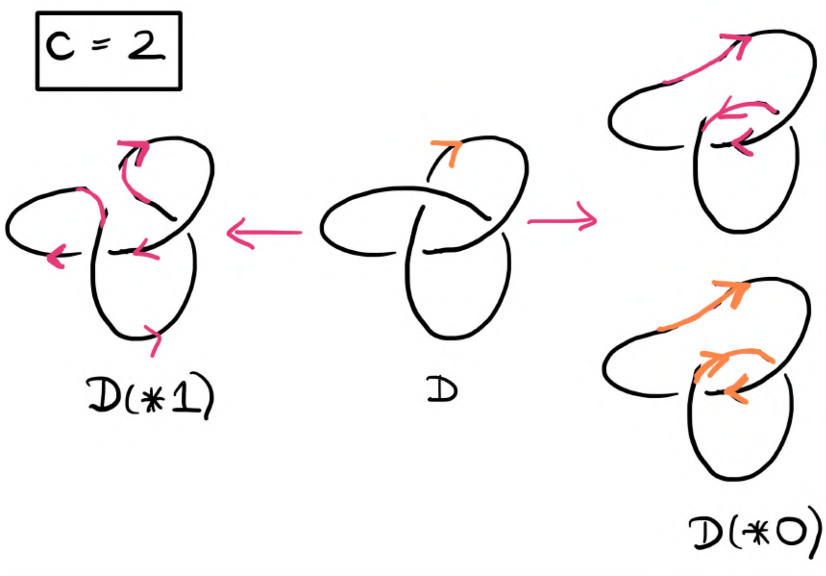


Figure 2.42

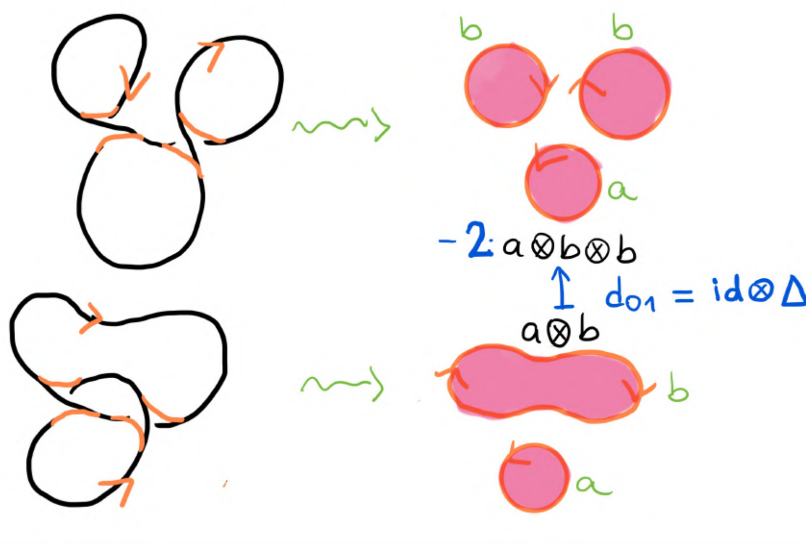


Figure 2.43

For the case where $D(*0)$ is a knot and $D(*1)$ is a link, one can see that this time $d_{0\to 1}$ maps the generator $\mathfrak{s}_o$ to $\mathfrak{s}_{\bar{o}}$ where o is an orientation on the knot $D(*0)$. In the example above, one can see this taking the right handed trefoil with the reversed orientation. Indeed $d_{0\to 1}$ is given by the merge map and $m'(a \otimes a) = a, m'(b \otimes b) = b$.
Hence $rank(d_{0\to 1}) = 2$. Then

$$2 \leq dim(Lee(D)) \leq dim(Lee(D(*0)) + dim(Lee(D(*1)) - 4 \leq 2 + 4 - 4 \leq 2$$

This concludes that if we have the result for $\leq c$ crossings then for a knot K with c crossings we have the result.
Now if L is a 2-component link, there are two possibilities:

- L is a disjoint union of two knot diagrams K_1, K_2
- L has a crossing such that resolving it gives to two knot diagrams.

In the first case, we have

$$Lee(L) = Lee(K_1) \otimes Lee(K_2)$$

Hence

$$4 \leq dim(Lee(L)) \leq 2 + 2$$

In the second case we get as before, since the two knots have less than c crossings,

$$4 \leq \dim(Lee(L)) \leq \dim(Lee(D(*0)) + \dim(Lee(D(*1)) \leq 2 + 2$$

Finally if L is a n component link and if the claim is true for link with less components, then we can resolve a crossing in L so that we get two link with $n-1$ components, hence

$$2^n \leq dim(Lee(L)) \leq 2^{n-1} + 2^{n-1} = 2^n$$

□

2.2.2 Khovanov-Lee spectral sequence

In Chapter 1, we saw how one can associate to a filtered chain complex of vector spaces a filtration spectral sequence. Recall that this filtration sequence has E_0 term the associated graded vector space and that it converges to the homology of the original filtered complex.

Lemma 2.10. *The associated graded vector space of $CLee(D)$ is $CKh(L)$.*

Proof. By definition of the associated graded vector space, the space $CLee^n(D)$ decomposes as graded vector space

$$CLee^n(D) = \bigoplus_{p \in \mathbb{Z}} G^p(CLee^n(D))$$

where

$$G^p CLee^n := F_p CLee^n / F_{p+1} C^n$$

$$F_p CLee^n / F_{p+1} CLee^n = \langle \{x \in CLee^n(D) \mid gr_f(x) = p\} \rangle$$

And since

$$CKh^{n,p}(D) = \langle \{x \in CKh^p(D) \mid gr_q(x) = p\} \rangle$$

We have

$$G^p CLee^n(D) \cong CKh^{n,p}(D)$$

□

Proposition 2.8. *The filtration spectral sequence associated to* $(CLee(D), F^*(CLee(D)))$ *has for* E_1 *term Khovanov homology* $Kh(L)$ *and* E_∞ *term Lee homology* $Lee(L)$*. Moreover each term of the spectral sequence is a link invariant.*

Proof. We know by Chapter 1 how the differentials of this spectral sequence are defined. Since we have

$$d' = d + \phi$$

and since the term E_0 is $CKh(D)$ and the differential is given by

$$d_0 = d$$

By definition of the filtration spectral sequence, the E_1 term of this spectral sequence is $Kh(L)$. Moreover since we have a bounded spectral sequence, we know that the filtration spectral sequence converges to the homology of the original filtered complex which is $Lee(L)$.
For the invariance of each term, take two diagrams of the same link and use the maps between the complexes $CLee(D), CLee(D')$ which are filtered. They induce isomorphism on the first page since they correspond to maps ρ_i* of the invariance of Khovanov homology. By 1.22, we are done. □

We call this spectral sequence Khovanov-Lee spectral sequence.

Proposition 2.9. *The differentials* d^r *in the Khovanov-Lee spectral sequence vanish if* $r \neq 0 \pmod 4$.

Proof. One way to see this is to use the fact that even though $(CLee(D), d')$ is not a $\mathbb{Z}$ graded complex, it is a $\mathbb{Z}/4\mathbb{Z}$-graded complex. Indeed d' shifts the filtration only by 0 or by 4. So $\pmod 4$, it is degree preserving. It is to say that one can decompose the complex as a direct sum of complexes

$$CLee(D) = CLee_0(D) \oplus CLee_1(D) \oplus CLee_2(D) \oplus CLee_3(D)$$

So the Khovanov-Lee spectral sequence decomposes into a direct sum of four spectral sequences of graded vector spaces. For example below are some pages of the Khovanov-Lee spectral sequence and in red is shown the sub spectral sequence corresponding to $CLee_0(K)$

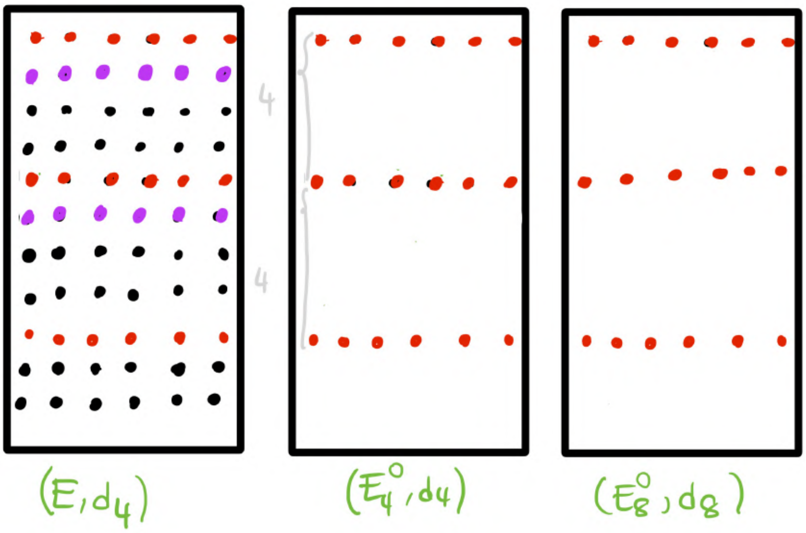


Figure 2.44

For each spectral sequence, the only non trivial differentials are on pages $4k$ for some $k \in \mathbb{N}$ for geometric reasons as is shown below

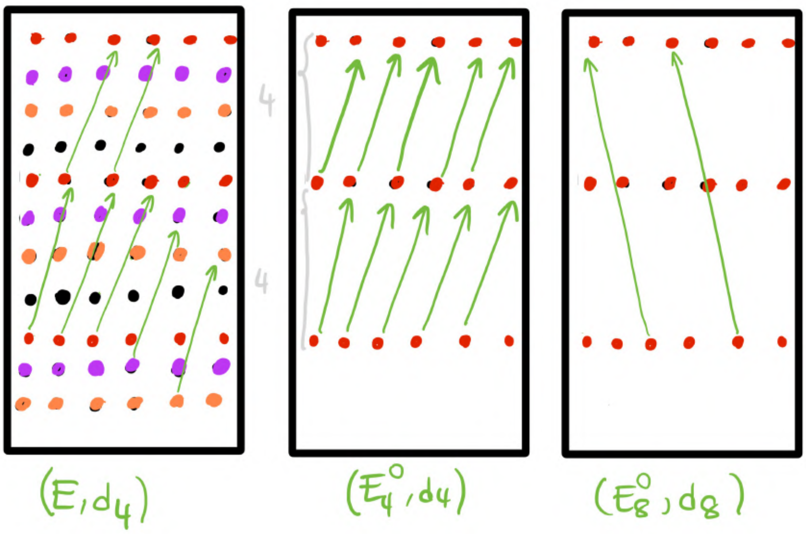


Figure 2.45

Hence the Khovanov-Lee spectral sequence does not admit any non trivial differential d^r for $r \neq 0$ mod4. $\square$

Since the differential on the page E_r lifts the q grading by r, the only interesting pages in this spectral sequences are E_0, E_4, E_8, E_{12}.... Indeed since the differentials on the pages

$E_i, i \neq 0 \pmod 4$ vanish,

$$\begin{aligned} E_1 = E_2 = E_3 = E_4 \\ E_5 = E_6 = E_7 = E_8 \\ \dots \end{aligned}$$

2.3 The Rasmussen invariant

Let K be a knot. Then we know by Lee's structure theorem that as ungraded vector spaces,

$$Lee(K) = \mathbb{Q} \oplus \mathbb{Q}$$

If we think of $Lee(K)$ in terms of the limit of the Khovanov-Lee spectral sequence then it means that there are only two surviving generators in $CLee(K)$ to E_∞ page and we note $s_{min}(K)$ and $s_{max}(K)$ the filtration gradings of these two generators. Indeed it is very natural to ask if this would give us a knot invariant!

Definition 2.8. *Let K be a knot. Define*

$$\begin{aligned} s_{max}(K) &:= \max(gr_f([x]) \mid [x] \in Lee(K), [x] \neq 0) \\ s_{min}(K) &:= \min(gr_f([x]) \mid [x] \in Lee(K), [x] \neq 0) \end{aligned}$$

Here are some remarks to prevent some confusions which may arise:

Remark 2.10. *Attention, $s_{min}(K)$ is not the smallest quantum degree which appears in $CLee(K)$. If a class $[x] \in CLee(K)$ realizes $s_{min}(K)$, it can be represented by some other $y \in CLee(K)$ having smaller filtration grading. However $s_{max}(K)$ corresponds to the biggest filtration grading appearing in $CLee(K)$ and which is not trivial in homology.*

Remark 2.11. *Even though the Rasmussen invariant is not defined by the spectral sequence itself, this spectral sequence gives vital information about $s_{min}(K)$ and $s_{max}(K)$ since it determines which terms survive to the E_∞ page, hence determines what are the surviving quantum gradings.*

In other words, by definition of $gr_f([x])$, $s_{min,max}(K)$ correspond to the filtration gradings of non zero homology classes of the two surviving generators in $Lee(K)$.

Proposition 2.10.

$$\begin{aligned} s_{min}(K) &= min(i \in \mathbb{Z} \mid F_i(Lee(K)) \neq F_{i+1}(Lee(K))) \\ s_{max}(K) &= max(i \in \mathbb{Z} \mid F_i(Lee(K)) \neq F_{i+1}(Lee(K)) \end{aligned}$$

Proof. Recall that the relation between the quantum grading and the filtration on Lee complex is given by

$$gr_f([x]) = \max(i \in \mathbb{Z} \mid x \in F_i Clee(K))$$

Moreover

$$gr_q(x) = i \leftrightarrow x \in F_i CLee, x \notin F_{i+1} CLee$$

So

$$\begin{aligned} s_{max}(K) &= \max(gr_q([x]) \mid \exists [x] \in Lee, [x] \neq 0) \\ &= \max(i \in \mathbb{Z} \mid \exists [x] \in F_i Lee, [x] \notin F_{i+1} Lee) \quad = \max(i \in \mathbb{Z} \| \ F_i Lee \neq F_{i+1} Lee) \end{aligned}$$

Similarly,

$$\begin{aligned} s_{min}(K) &= \min(gr_q([x]) \mid \exists [x] \in Lee, [x] \neq 0) \\ &= \min(i \in \mathbb{Z} \mid \exists [x] \in F_i Lee, [x] \notin F_{i+1} Lee) \quad = \min(i \in \mathbb{Z} \| \ F_i Lee \neq F_{i+1} Lee) \end{aligned}$$

□

Proposition 2.11. *Let K be a knot.Then $s_{min}(K)$ and $s_{max}(K)$ are odd.*

Proof. By 2.1 for a knot all quantum gradings are odd. Since $s_{min}(K)$ and $s_{max}(K)$ are given respectively as maximum and minimum quantum gradings of non zero (Lee) homology classes, they are odd. □

Proposition 2.12. *$s_{min}(K)$ and $s_{max}(K)$ are knot invariants.*

Proof. Let K and K' be equivalent knots. Taking the maps ρ_i' which are filtered isomorphisms we must have that the minimum and the maximum integer i such that respectively $F_i(Lee(K) \neq F_{i+1}Lee(K)$ and $F_i(Lee(K') \neq F_{i+1}Lee(K')$ should be the same for the two knots. □

Definition 2.9. *Let K be a knot. We define the Rasmussen invariant of K by*

$$s(K) := \frac{s_{min}(K) + s_{max}(K)}{2}$$

Moreover, $s_{min}(K)$ and $s_{max}(K)$ are not at all independant. They determine each other. We want to show

$$s_{min}(K) = s_{max}(K) - 2$$

for any knot K.

Proposition 2.13. *Let K be a knot and let D be a diagram with orientation o and note r the oriented resolution. Then the cycles $\mathfrak{s}_o \pm \mathfrak{s}_{\bar{o}}$ are contained in different $\mathbb{Z}/4\mathbb{Z}$ quantum grading summands in $CLee(D)$.*

Proof. As we observed before, $\mathfrak{s}_o$ and $\mathfrak{s}_{\bar{o}}$ are obtained by the same resolved diagram modulo orientations. We define an involution

$$\begin{aligned} T : & A' \to A' \\ & 1 \mapsto -1 \\ & X \mapsto X \end{aligned}$$

Then extend T to $A'^{\otimes |D_r|}$:

$$T(x \otimes y) = T(x) \otimes T(y)$$

Now observe:

(1) T acts as identity on the generators containing an even number of 1's in the tensor product. Denote $CLee_{even}(D)$ the span of those generators.

(2) T acts as $-id$ on those having an odd number of 1's, let them span $CLee_{odd}(D)$.

Now recall that the quantum grading of any homogeneous generator in $CLee(D)$ is given by

$$gr_q(x) = gr_h(x) + deg(x) + n_+ - n_-$$

Moreover the number of terms in the tensor product is fixed to $|D_r|$ and $deg(x) = \#1 - \#X$. Moreover $gr_h(x) = 0$ for any x so we get

$$gr_q(x) = 2\#1 - |D_r| + n_+ - n_-$$

and since $n_+ - n_-$ is also fixed, modulo 4 we have

$$gr_q(x) = -|D_r| + n_+ - n_- mod(4)$$

$\forall x \in CLee_{even}(D)$ and

$$gr_q(x) = 2 - |D_r| + n_+ - n_- mod(4)$$

$\forall x \in CLee_{odd}(D)$.
Hence, $CLee_{even}$ and $CLee_{odd}$ live in different quantum gradings.

Moreover

$$T(a) = b \qquad\qquad T(b) = a$$

where $a = 1 + X$ and $b = X - 1$. We have

$$T(\mathfrak{s}_o) = \mathfrak{s}_{\bar{o}}$$

so

$$T(\mathfrak{s}_o + \mathfrak{s}_{\bar{o}}) = \mathfrak{s}_o + \mathfrak{s}_{\bar{o}}$$
$$T(\mathfrak{s}_o - \mathfrak{s}_{\bar{o}}) = -\mathfrak{s}_o + \mathfrak{s}_{\bar{o}}$$

We conclude that

$$\mathfrak{s}_o + \mathfrak{s}_{\bar{o}} \in CLee_{even}(D)$$
$$\mathfrak{s}_o - \mathfrak{s}_{\bar{o}} \in CLee_{odd}(D)$$

□

Corollary 2.4. $s_{max}(K) > s_{min}(K)$

Proof. We found that for a knot, only odd quantum degrees survive and we just showed that there exist two non zero homology classes in $Lee(K)$ whose quantum degrees are different. This concludes

$$s_{max}(K) > s_{min}(K)$$

□

Corollary 2.5. $s_{min}(K) = gr_q([\mathfrak{s}_o]) = gr_q([\mathfrak{s}_{\bar{o}}])$

Remark 2.12. *$Lee(K)$ is a graded $\mathbb{Q}$-vector space under the grading gr_4 defined as gr_q (mod 4) and is generated by α, β of gradings $s_{min}(K), s_{max}(K)$.*

Proof. Denote $\alpha = [\mathfrak{s}_o + \mathfrak{s}_{\bar{o}}], \beta = [\mathfrak{s}_o - \mathfrak{s}_{\bar{o}}]$. By the previous discussion α or β generate the summand with grading $s_{min}(K)$. Moreover

$$[\mathfrak{s}_o] = \frac{1}{2}\alpha + \frac{1}{2}\beta$$
$$[\mathfrak{s}_{\bar{o}}] = \frac{1}{2}\alpha - \frac{1}{2}\beta$$

So

$$\begin{aligned} gr_f([\mathfrak{s}_{\bar{o}}]) = gr_f([\mathfrak{s}_{\bar{o}}]) &= min(gr_f(\alpha), gr_f(\beta)) \\ &= s_{min}(K) \end{aligned}$$

□

Remark 2.13. *We also have* $s_{max}(K) = \max(gr_f(\alpha), gr_f(\beta))$

Properties of the Rasmussen invariant

Proposition 2.14. *Let K be a knot and $\bar{K}$ its mirror image. Then*

$$s(\bar{K}) = -s(K)$$

Proof. We will show that

$$\begin{aligned} s_{max}(\bar{K}) &= -s_{min}(K) \\ s_{min}(\bar{K}) &= -s_{max}(K) \end{aligned}$$

Now similarly to Khovanov homology case, as filtered complexes we have

$$Lee(\bar{K}) \cong (Lee(K))^*$$

By 1.3 , the filtration spectral sequences associated to these dual filtered complexes should also be dual, so the generator in $Lee(K))$ with filtration grading $s_{min}(K)$ is sent to the generator with filtration grading $-s_{min}$. Similarly, the one with filtration grading $s_{max}(K)$ is sent to $-s_{max}(K)$ so we get

$$\begin{aligned} s(\bar{K}) &= \frac{-s_{min}(K) - s_{max}(K)}{2} \\ &= -s(K) \end{aligned}$$

□

Lemma 2.11. *Let K_1, K_2 be two knots. There is a short exact sequence*

$$0 \to Lee(K_1 \# K_2) \xrightarrow{p^*} Lee(K_1) \otimes Lee(K_2) \xrightarrow{m^*} Lee(K_1 \# K_2) \to 0$$

where p^*, m^* are filtered of degree -1.

Proof. Recall that by proposition 2.3 we have a short exact sequence at the chain level

$$0 \to CLee(K_1 \# K_2)\{2\} \xrightarrow{i} CLee(K_1 \# K_2^r) \xrightarrow{p} CLee(K_1 \sqcup K_2)\{1\} \to 0$$

We obtain the long exact sequence at the level of Lee homology:

$$\cdots \to Lee(K_1 \# K_2^r) \to Lee(K_1 \# K_2) \to Lee(K_1 \sqcup K_2) \to Lee(K_1 \# K_2^r)$$

This long exact sequence is encoded in the exact triangle below

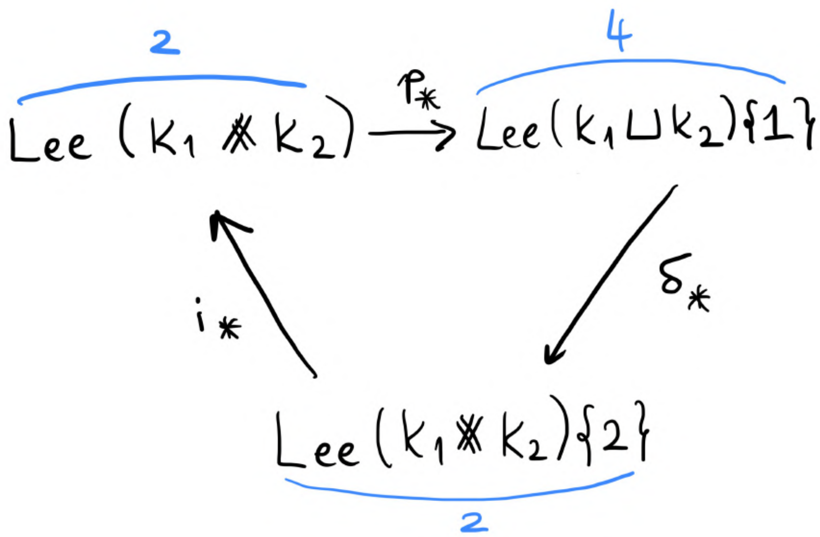


Figure 2.46: An exact triangle

Now using the Lee structure theorem and the lemma 1.1 we get $i_* = 0$ and hence we get the short exact sequence

$$0 \to Lee(K_1 \# K_2) \xrightarrow{p_*} Lee(K_1) \otimes Lee(K_2)\{1\} \xrightarrow{\partial_*} Lee(K_1 \# K_2)\{2\} \to 0$$

Since normally the maps ∂ and p are filtered, if we forget about the shiftings they become filtered of degree -1. $\square$

Proposition 2.15. *Let K_1, K_2 be two knots. Then*

$$s(K_1 \# K_2) = s(K_1) + s(K_2)$$

Proof. By the previous lemma we do have:

$$\to Lee(K_1 \# K_2) \xrightarrow{p^*} Lee(K_1) \otimes Lee(K_2) \xrightarrow{\partial^*} Lee(K_1 \# K_2) \to 0$$

Let $\mathfrak{s}_a^i, \mathfrak{s}_b^i$ be the canonical generators according to their label on the component where the connected sum occurs. We claim that $Lee(K_1\#K_2)$ has a canonical generator $[\mathfrak{s}_o]$ which is getting mapped to $[\mathfrak{s}_a^1 \otimes \mathfrak{s}_b^2]$ under p^*. Then

$$\begin{aligned} gr_q([\mathfrak{s}_a^1 \otimes \mathfrak{s}_b^2]) &\geq gr_q(\rho'_*([\mathfrak{s}_o])) \\ &\geq gr_q([\mathfrak{s}_o]) - 1 \\ &\geq s_{min}(K_1\#K_2) - 1 \end{aligned}$$

$$\begin{aligned} s_{min}(K_1\#K_2) - 1 &\leq gr_q(\mathfrak{s}_a^1) + gr_q(\mathfrak{s}_b^2) \\ &\leq s_{min}(K_1) + s_{min}(K_2) \end{aligned}$$

Now apply to $\bar{K_1}$ and to $\bar{K_2}$:

$$\begin{aligned} &-s_{max}(K_1\#K_2) - 1 \leq -s_{max}(K_1) - s_{max}(K_2) \\ \Rightarrow & s_{max}(K_1\#K_2) + 1 \geq s_{max}(K_1) + s_{max}(K_2) \\ \Rightarrow & s_{min}(K_1\#K_2) + 3 \geq s_{min}(K_1) + s_{min}(K_2) + 4 \\ \Rightarrow & s_{min}(K_1\#K_2) \geq s_{min}(K_1) + s_{min}(K_2) + 1 \end{aligned}$$

So

$$s_{min}(K_1\#K_2) = s_{min}(K_1) + s_{min}(K_2) + 1$$

Similarly

$$s_{max}(K_1\#K_2) = s_{max}(K_1) + s_{max}(K_2) - 1$$

Hence

$$s(K_1\#K_2) = s(K_1) + s(K_2)$$

□

Proposition 2.16. *Let K be a knot. Then*

$$s_{min}(K) = s_{max}(K) - 2$$

Proof. In the last lemma, take $K_1 = K$ and $K_2 = U$. Then we get a short exact sequence

$$0 \to Lee(K) \xrightarrow{p*} Lee(K) \otimes Lee(U) \xrightarrow{m*} Lee(K) \to 0$$

We have $Lee(U) = \mathbb{Q}\alpha \oplus \mathbb{Q}\beta$ and one of $\mathfrak{s}_o, \mathfrak{s}_{\bar{o}}$ has label a on the connected component where # occurs, call it $\mathfrak{s}_a$ and call the other $\mathfrak{s}_b$. We have

$$gr_f([\mathfrak{s}_a] + \varepsilon[\mathfrak{s}_b]) = s_{max}(K)$$

for an $\varepsilon \in \{\pm 1\}$. Then we have

$$m^*([\mathfrak{s}_a] + \varepsilon[\mathfrak{s}_b] \otimes [a]) = [\mathfrak{s}_a]$$

Moreover since $(m)^*$ is filtered of degree -1 it can only decrease the filtration grading by at most 1:

$$s_{max}(K) - 1 \leq gr_f([\mathfrak{s}_a + \varepsilon \mathfrak{s}_b \otimes a]) \leq gr_f([\mathfrak{s}_a]) + 1$$

Then since $gr_f(a) = min(gr_q(1), gr_q(X)) = -1$ and $s_{max}(K) = max(gr_q([\mathfrak{s}_a + \mathfrak{s}_b]), gr_q([\mathfrak{s}_a - \mathfrak{s}_b]))$ we have

$$s_{max}(K) - 1 \leq s_{min}(K) + 1$$
$$\Rightarrow s_{max}(K) \leq s_{min}(K) + 2$$

The other inequality is by 2.4 and the fact that $s_{min}(K)$ and $s_{max}(K)$ are odd,

$$s_{max}(K) \geq s_{min}(K) + 2$$

□

In particular, using the basis a, b of A' and the canonical generators we showed that the homology classes of $s_{min}(K)$ or $s_{max}(K)$ determine $s(K) = s_{min}(K)+1 = s_{max}(K)-1$ which may simplify computation of the invariant s.

Our next goal is to show that s induces a group homomorphism

$$s : Conc^3 \to \mathbb{Z}$$

For this, we will use the relation of Khovanov homology with cobordisms between two links. Recall that if L_0, L_1 are two link and if S is a cobordism from L_0 to L_1, then Khovanov homology induces a graded map

$$\phi_S : Kh(L_0) \to Kh(L_1)$$

of quantum degree $\chi(S)$. This map was defined for Khovanov homology by composing "elementary maps" defined either by Reidemeister moves or Morse moves. In defining them we used the maps ρ_i and $m, \Delta, \iota, \epsilon$. To get a filtered map of filtered degree $\chi(S)$:

$\phi' : Lee(L_0) \to Lee(L_1)$ on Lee homology, one can take the corresponding maps in Lee homology, namely ρ_i' and $m', \Delta', \iota, \epsilon$.

Definition 2.10. *Let S be a cobordism from two oriented links L_0 to L_1 and o_S an orientation on S. We say that that the orientations o_0, o_1 are compatible with o_S if the orientations induced on L_0 and L_1 by o satisfies:*

$$o_0 = \bar{o_S}$$
$$o_1 = o_S$$

More generally, we say that a pair of orientations (o_0, o_1) respectively on L_0, L_1 are compatible if there exists an orientation on S such that they are compatible with o_S.

Proposition 2.17. *Let S be a cobordism from L_0 to L_1 with no closed components and let ϕ_S be the map induced on Lee homologies. If o is an orientation on L_0 then*

$$\phi_S'([\mathfrak{s}_o]) = \sum_{o_I} a_I [\mathfrak{s}_{o_I}]$$

where o_I runs over all orientations on L_0 which are L_1-compatible with o and where each a_I is non zero.

Proof. The proof is by induction on the number of elementary moves in the movie representation of S.
Case n=1:
There is only one move, hence L_0 and L_1 differ by an elementary move. We treat each move separately:

- Morse moves: For the Morse moves we show that it holds at the chain level.
  - 0-handle move: We have $\phi = id \otimes \iota$ hence

$$\begin{aligned}\phi(\mathfrak{s}_o) &= id \otimes \iota((\mathfrak{s}_o \otimes 1)) \\ &= \mathfrak{s}_o \otimes \iota(1) \\ &= \mathfrak{s}_o \otimes \frac{a-b}{2}\end{aligned}$$

    Since L_1 is obtained by adding a circle to L_0 there are two orientations on L_1 compatible with o and the corresponding states are:

$$\mathfrak{s}_{o_1} = \mathfrak{s}_o \otimes a$$
$$\mathfrak{s}_{o_2} = \mathfrak{s}_o \otimes b$$

- 1-handle move: We separate the two cases where a component of L_0 may split into two compponents in L_1 or two components merge into one in L_1

  Splittings case: In this case there is a unique o-compatible orientation on L_1, which is taking o on $D_1 \setminus D'$ where D' is one of the new components and taking the induced orientation on D'. Moreover by definition the strands in the neighbourhood where the the move happens have opposite orientations so by 2.5 they have the same labels. This implies that $\mathfrak{s}_{o_I}$ will be of the form $\mathfrak{s}_o \otimes a$ if the new circle has a label a and it will be $\mathfrak{s}_0 \otimes b$ otherwise (taking a particular ordering of circles). Also, since $\Delta'(a) = a \otimes a, \Delta'(b) = b \otimes b$, we clearly have

$$\phi(\mathfrak{s}_o) = \mathfrak{s}_{o_I}$$

  Merging case: Now suppose that L_1 was obtained by merging two components in L_0 into one component. If the two strands in a neighbourhood where the move happens are oriented oppositely under o, then there will be a unique orientation o_1 on L_1 compatible with o. Moreover the labels are same still by 2.5. Using $m'(a \otimes a) = 2a$ and $m'(b \otimes b) = -2b$ hence we get $\phi(\mathfrak{s}_o) = \pm 2\mathfrak{s}_I$. If the two strands have parallel orientations then o cannot be extended to an orientation on S, so there is no compatible orientation on L_1. Moreover the labels must be the different and we use $m'(a \otimes b) = m'(b \otimes a) = 0$ to conclude.

- 2-handle move: Now we have $L_0 = L_1 \sqcup U$ and hence on L_1 there is a unique orientation compatible with o which is its restriction on L_1. Knowing that ϕ' is given by $id \otimes \varepsilon'$. The circle in L_0 can be labeled a or b and in either case

$$\begin{aligned}\phi(\mathfrak{s}_o) &= id \otimes \varepsilon'(\cdots \otimes a)\\ &= id(\cdots) \otimes 1\\ &= \mathfrak{s}_{o_I}\end{aligned}$$

  or

$$\begin{aligned}\phi(\mathfrak{s}_o) &= id \otimes \varepsilon'(\cdots \otimes b)\\ &= id(\cdots) \otimes 1\\ &= \mathfrak{s}_{o_I}\end{aligned}$$

- Reidemeister moves:

  For Reidemeister moves we use 2.6 with the fact that for two links which are related by a Reidemeister move there is a unique compatible orientation:

  Lemma 2.12. *Suppose L and $\tilde{L}$ are related by the ith Reidemeister move. Let $\tilde{o}$ be the orientation on $\tilde{L}$ by o. Then $\rho'_{i*}([\mathfrak{s}_o])$ is a non zero multiple of $[\mathfrak{s}_{\tilde{o}}]$.*

We conclude that the proposition holds for elementary cobordisms.
Now suppose that it is true for a movie with k moves for $k \leq n-1$. Let S be a movie with n moves. We can decompose S into two cobordisms:

$$S_1 : L_0 \to L_{\frac{1}{2}}$$
$$S_2 : L_{\frac{1}{2}} \to L_1$$

Then we have

$$\phi_S = \phi_{S^2} \circ \phi_{S^1}$$

Hence

$$\begin{aligned}
\phi_S([\mathfrak{s}_o]) &= \phi_{S_2}(\phi_{S_1}([\mathfrak{s}_o]) \\
&= \phi_{S_2}(\sum_{o_I} a_I[\mathfrak{s}_{o_I}]) \\
&= \sum_{o_I} a_I \phi_{S_2}([\mathfrak{s}_{o_I}]) \\
&= \sum_{o_J}\sum_{o_I} a_I a_J [\mathfrak{s}_{o_J}]
\end{aligned}$$

Now it suffices to show that the pairs (o_I, o_J) where o_I is an orientation on $L_{\frac{1}{2}}$ compatible with o and o_J an orientation on L_1 compatible with o_I are in bijection with the set of orientations o_K on L_1 compatible with o.
If (o_I, o_J) is such a pair, then there is an orientation on S compatible with the pair. We take o_K to be o_J. Since we glue the two cobordisms on the component $L_{\frac{1}{2}}$ by an orientation-reversing homeomorphism this determines an orientation on S compatible with (o, o_K). The other direction is even simpler. For if o_K is compatible with o, then o and o_I are compatible where o_I is the orientation of S restricted to $L_{\frac{1}{2}}$. The induced orientation on the second part of the cobordism gives a compatible pair $(\bar{o}_I, o_K)$. These two are inverses one of the other and this completes our proof. □

Remark 2.14. *This proposition holds only at homological level! Indeed, the maps ρ'_i define isomorphisms only at homological level.*

Corollary 2.6. *If S is a connected cobordism (smoothly embedded) between knots K_0 and K_1 then ϕ_S is an isomorphism.*

Proof. Since for a knot K the only generators of $Lee(K)$ are $[\mathfrak{s}_o]$ and $[\mathfrak{s}_{\bar{o}}]$ we get

$$\begin{aligned}\phi_S([\mathfrak{s}_o]) &= \sum_{o_I} a_I[\mathfrak{s}_{o_I}] \\ &= a_1[\mathfrak{s}_{o_1}] + \bar{a_1}[\mathfrak{s}_{\bar{o_1}}]\end{aligned}$$

But a_1 or $\bar{a_1}$ should be zero since only one of them can be compatible with $\mathfrak{s}_o$, by connectedness. Without loss of generality suppose that $a_I \neq 0$. Then

$$\phi_S(\{[\mathfrak{s}_o], [\mathfrak{s}_{\bar{o}}]\}) = \{[\mathfrak{s}_{o_1}], [\mathfrak{s}_{\bar{o_1}}]\}$$

So ϕ_S send one basis onto another basis so ϕ_S is an isomorphism. □

We can finally prove the following result which connects the Rasmussen invariant to the sliceness of a knot.

Theorem 2.6. *Let K be a knot. Then*

$$|s(K)| \leq 2g_*(K)$$

where g_* is the smooth slice genus of K.

Proof. Let K be a knot. Let S be an orientable connected cobordism between K and U in $\mathbb{R}^3 \times [0,1]$ such that $\chi(S) = -2g$. Let ϕ'_S be the induces map

$$\phi'_S : Lee(K) \to Lee(U)$$

Let $x \in Lee(K) \setminus \{0\}$ be a class such that $gr_f(x) = s_{max}(K)$. Then $\phi'_S(x)$ is not zero in $Lee(U)$ since ϕ'_S is an isomorphism. We also know that ϕ'_S is a filtered map of degree $\chi(S)$, hence

$$gr_f(\phi'_S) \geq gr_f(x) - 2g$$

Since $s_{max}(U) = 1$ we have,

$$gr_f(\phi'_S(x)) \leq 1$$

Combining the two inequalities we get

$$gr_f(x) = s_{max}(K) \leq 2g + 1$$

Since $s_{max}(K) - 1 = s(K)$, we have

$$s(K) \leq 2g$$

Now to show $-s(K) \leq 2g$, take the mirror image $\bar{K}$ of K, bounding a surface $m(S) \cong S$. We apply the same arguments and use $s(\bar{S}) = -s(K)$ so

$$-s(K) \leq 2g$$

So for any surface S with genus g such that $\partial S = K$, we have

$$|s(K)| \leq 2g$$

In particular this holds for the slice genus of the knot. $\square$

Since s gives a lower bound for the slice genus just like the signature, in can be used to tell if a knot is not slice. Indeed if K is slice then $s(K)$ must be zero.

Theorem 2.7. *$s : \{Knots\} \to \mathbb{Z}$ induces a group homomorphism*

$$Conc(S^3) \to \mathbb{Z}$$

Proof. Let K_1, K_2 be two concordant knots. Then $K_1 \# \bar{K}_2^r$ is slice. Then

$$s(K_1 \# \bar{K}_2^r) = s(K_1) - s(K_2)$$

hence s is well defined. Moreover

$$\begin{aligned} s([K_1 \# K_2]) &= s(K_1 \# K_2) \\ &= s(K_1) + s(K_2) \\ &= s([K_1]) + s([K_2]) \end{aligned}$$

and

$$\begin{aligned} s([\bar{K}_1^r]) &= s(\bar{K}_1^r) \\ &= -s(K_1^r) \\ &= -s([K_1]) \end{aligned}$$

$\square$

Corollary 2.7. *Let D and D' be two knot diagrams that differ only by a single crossing.*

Then

$$|s(K) - s(K')| \leq 2$$

Proof. Consider the following movie between two such skein knots

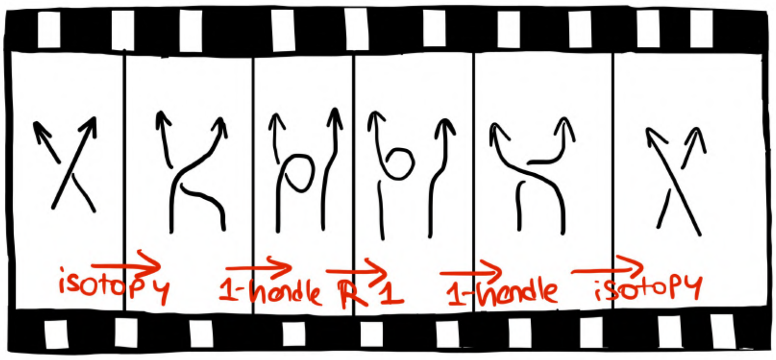


Figure 2.47: A movie from K_+ to K_-

Since the corresponding map $\phi' : Lee(K_+) \to Lee(K_-)$ is of filtered degree $\chi(S) = -2$ we have that $s_{min}(K_+)$ and $s_{min}(K_-)$ can differ at most by 2. $\square$

Remark 2.15. *In particular, we have*

$$s(K) - s(K') \in \{-2, 0, +2\}$$

and there are examples of knots where s doesn't change.

Proposition 2.18. *Let $u(K)$ be the unknotting number of a knot K. Then we have*

$$|s(K)| \leq 2u(K)$$

Proof. Denote $n = u(K)$, then let K_i correspond to knot diagrams such that K_i and K_{i+1} differ only by a crossing, using the above lemma with triangular inequality,

$$\begin{aligned} |s(K) - s(U)| &= |s(K) - s(K_1) + s(K_1) - s(K_2) + \cdots + s(K_n) - s(U)| \\ &\leq 2u(K) \end{aligned}$$

$\square$

In particular we get $|s(T_{p,q})| \leq 2u(T_{p,q})$

Notice that the signature of a knot verifies the same property. One can ask if s is stronger then σ for the bound of the unknotting number. The answer is yes. Indeed in general σ is smaller then s.

Positive knots

Proposition 2.19. *Let K be a positive knot. Then we have*

$$s(K) = g_*(K) = g(K)$$

Proof. Let D be a positive diagram for K with n crossings. Then since the oriented resolution consists of only 0-resolutions and it lives in the extreme corner of the cube of resolutions. Since there is no smaller homology degree since we are at the smallest possible quantum degree, a class is represented by its unique representative. This gives

$$s(K) = gr_q(\mathfrak{s}_o) + 1$$

and

$$gr_q(\mathfrak{s}_o) = -k + n$$

where k is the number of circles in the resolved diagram of the oriented resolution.
Moreover the Seifert algorithm gives a Seifert surface S for K with Euler characteristic $k - n$ by 1.4, thus

$$2g(K) \leq 2g(S) = n - k + 1 \leq s(K) \leq 2g_*(K)$$

But since we also have $g_*(K) \leq g(K)$, we get equalities

$$s(K) = 2g_*(K) = 2g(K)$$

□

The Milnor conjecture now is a direct consequence since we know $g(T_{p,q})$.

Corollary 2.8 (Milnor Conjecture). $g_*(T_{p,q}) = \frac{(1-p)(1-q)}{2}$

Moreover we get the unknotting number of the torus knots by the following corollary.

Corollary 2.9. *We have* $u(T_{p,q}) = \frac{(1-p)(1-q)}{2}$

Proof. We already claimed in Chapter 1 that

$$u(T_{p,q}) \leq \frac{(p-1)(q-1)}{2}$$

Now we have $\frac{(p-1)(q-1)}{2} \leq u(T_{p,q})$. So we recover 1.26

$$u(T_{p,q}) = \frac{(p-1)(q-1)}{2}$$

□

We saw that for a positive knot K with a positive diagram D, the genus of K is given by

$$2g(k) = n(D) - O(D) + 1$$

where n is the number of crossings in D and $O(D)$ is the number of Seifert circles.

One can easily show that this equality can be generalized as an inequality to arbitrary knots.

Proposition 2.20. *Let K be a knot and D be a diagram for K. Then we have*

$$s(K) \geq w(D) - O(D) + 1$$

Proof. We use induction on the number of negative crossings of the diagram. If all the crossings are positive, then $s(K) = w(D) - O(D) + 1$ holds by the proposition. If a positive crossing is changed into a negative one, then the right hand side decreases by 2, and the left hand side decreases at most by 2 by 2.7. So the inequality is preserved. □

Alternating knots We return back to Bar Natan's conjecture REF now to establish the connection between the signature and the Rasmussen invariant:

Proposition 2.21. *Let K be a knot such that the Khovanov-Lee spectral sequence converges after the E_4 term. Then $s_{BN}(K) = s(K)$.*

Proof. We observe that the Khovanov homology groups of bidegree difference (1,4) are paired except at bidegrees $(0, s_{max}(K)), (0, s_{min}(K))$. Indeed if $d^i = 0$ for $i \geq 5$ then E_5 page terms are the homology of Khovanov homology groups with respect to the map d_4 which gives Lee homology groups. Moreover d_4 vanishes on $Kh^{a,b}(K)$ for all $(a, b) \neq (0, s_{min,max}(K)$. Then taking the homology of E_4 within d_4 the E_5 page is given by

$$Kh^{a,b}(K) \cong Kh^{a+1,b+4}, \forall (a, b) \neq (0, s_{max}(K)), (0, s_{min}(K))$$

This discussion is translated to the Poincaré polynomial by

$$\begin{aligned}
P_{Kh}(t,q) &= \sum_{r\in\mathbb{Z}} qdim(Kh^r(K))t^r \\
&= \sum_{n\in\mathbb{Z}}\sum_{r\in\mathbb{Z}} dim(Kh^{r,m}(K))t^r q^n \\
&= dim(Kh^{0,s_{min}(K)})q^{s_{min}(K)} + dim(Kh^{0,s_{max}(K)})q^{s_{max}(K)} + \sum_{r\in\mathbb{Z}}\sum_{n\neq s_{min,max}(K)} dim(Kh^{r,n})t^r q^n
\end{aligned}$$

Now call $A(t,q)$ the last polynomial above where the coefficients of the monomials $t^a q^b$ and $t^{a+1}q^{b+4}$ are the same for all $a \geq 1, b > s_{min}(K)$. In other terms A is of the form

$$\begin{aligned}
A(t,q) &= a_{0,0}(1+tq^4+t^2q^8+...) + a_{0,1}(q+tq^5+...) + a_{02}(q'2+tq^6+...)\cdots \\
&= (1+tq^4)Q(K)(t,q)
\end{aligned}$$

where $Q(K)$ is a Laurent polynomial in q,t with only positive coeffiecients depending only on the dimensions of the Khovanov invariants of the knot. So

$$P_{Kh(K)}(t,q) = q^{s(K)}(q+q^{-1}) + (1+tq^4)Q(K)$$

$\square$

It would be a good idea to look for general conditions under which a knot admits this type of Poincaré polynomial. This would make the computation of the Rasmussen invariant an affair of computing Khovanov homology.
A natural condition that comes to mind would be to make the differentials in the Khovanov-Lee spectral sequence vanish after the E_4 term so that the Poincaré polynomial has the form described above.

Definition 2.11. *Let K be a knot and let $\mu(K) = \{a/2-b \mid q^a t^b$ is a monomial in $P_{Kh}\}$. We define the (homological) width of K as one more than the difference between the maximum and the minimum elements in $\mu(K)$.*

Proposition 2.22. *If $W(K) \leq 3$, then the differential d_4 can be non vanishing, and higher differentials do vanish for geometric reasons as above. In other terms Khovanov-Lee spectral sequence converges after the E_4 term.*

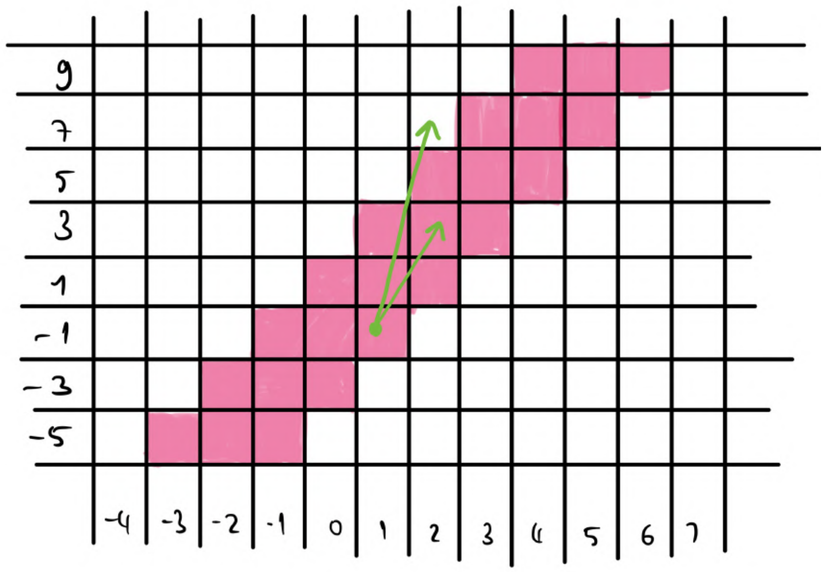


Figure 2.48: A knot with width $W(K) = 2$

Theorem 2.8. *If K is an alternating knot than $s(K) = \sigma(K)$.*

Proof. In [Lee02], Lee showed that for alternating knots we have $W(K) = 2$ and moreover Bar Natan's conjecture holds with $s(K) = \sigma(K)$. In particular by the proposition above this gives for an alternating knot K

$$s(K) = \sigma(K)$$

□

Remark 2.16. *Attention, one can easily see that the Rasmussen invariant is stronger than the signature invariant. Indeed consider the torus knots $T_{3,5}$ and $T_{4,5}$. Their signatures are both 8 but their Rasmussen invariants are respectively 8 and 12.*

Chapter 3

Some computations and applications of the Rasmussen invariant

In this chapter we mention informally some classical applications, computations and generalizations of the Rasmussen invariant to see some examples of situations where this, still pretty mysterious, invariant appears. Indeed, the number of families of knots for which we know to compute Rasmussen invariant is restricted. Those families include as we saw torus knots, positive and alternating knots aswell as what we call quasi-positive and quasi-alternative knots and homogeneous knots. In other cases, one has to deal with the general case in order to see if one can obtain the Rasmussen invariant easily from the Khovanov-Lee spectral sequence, by a certain bound or the Poincaré polynomial.

3.1 The Conway knot is not slice

As we saw, the Rasmussen invariant and the signature are concordance invariants and they give lower bounds on the slice genus, vanishing on the slice class in the concordance. They can be used to show that a knot is not slice. However, when these invariants vanish they do not give any information on the sliceness of the knot. A pathological situation is when all such known invariants vanish for a knot so that one cannot obstruct sliceness using them. For a long time, the Conway knot C below had been such an example. Its sliceness was not known until 2019.

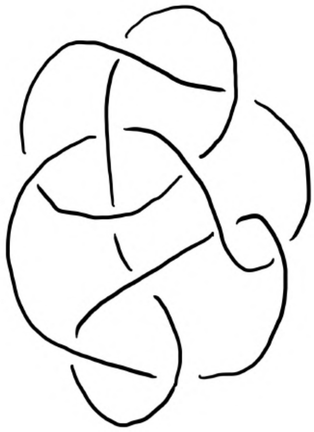


Figure 3.1: the Conway knot

In her PhD thesis [Pic20], Lisa Piccirillo showed that the Conway knot is not slice using the sliceness and the trace of a knot.

Definition 3.1. *Let K be a knot. The trace of K denoted $X(K)$ is a 4-dimensional manifold obtained by attaching a 0 framed 2-handle to the 4-ball B^4 along the knot K.*

The next lemma underlines the relation between the trace and the sliceness

Lemma 3.1 (Trace lemma)**.** *Let K be a knot. $g_*(K) = 0$ if and only if $X(K)$ smoothly embeds in S^4.*

In particular as a corollary, we have

Corollary 3.1. *if K and K' have diffeomorphic traces then, K is slice if and only if K' is slice.*

In her paper, Piccirillo found a knot K' such that $X(K') \cong X(C)$, and she showed that K' is not slice by showing that $s(C) = 2$. In this section we mention the calculation as a computational example.
She computed, using Bar Natan Fast Khovanov computations [BN1] the Poincaré polynomial

$$P_{Kh}(t,q) = \sum_{i,j} t^i q^j rank(Kh^{i,j}(K))$$

The plotted values of $rank(Kh^{i,j}(K)$ (over $\mathbb{Q}$) are as below

	-3	-2	-1	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14
49																		
47																		
45																		
43																		
41																		
39																		
37																		
35																		
33																		
31																		
29																		
27																		
25																		2
23																1	4	4
21															2	3	4	4
19														3	4	6	4	1
17												1	4	6	4	2	1	
15											2	4	5	3	2	1		
13										3	4	5	4	1				
11									3	5	4	2						
9							1	3	5	3	1							
7						2	3	3	3									
5					1	3	3	2										
3				1	3	3												
1			2	2	2													
-1		1		1														
-3		2																
-5	1																	

	15	16	17	18	19	20	21	22	23	24	25	26	27	28	29	30	31	32
49																		1
47																		
45															1	1	1	
43														1	1			
41												1	1	1	1			
39										1	1	1	2	1				
37									1	1	2	1						
35							1	1	3	2	1	1						
33						2	3	2	1	1								
31				1	2	3	2	2	1									
29			2	2	4	4	1											
27	1	2	4	4	2	1												
25	3	5	3	2	1													
23	3	4	2															
21	3	1																
19	1																	
17																		
15																		
13																		
11																		
9																		
7																		
5																		
3																		
1																		
-1																		
-3																		
-5																		

Figure 3.2: Table 4.1 in [Pic20]

Since we know that the Lee homology lives in 0-homological degree, we see that $s(K') \in \{0, 2\}$ since $s_{max}(K') \in \{3, 1\}$. We will show that

$$s(K') = 2$$

Here, we use the fact that all higher differentials in Khovanov-Lee spectral sequence are filtered of degree $4n$. Consider a generator $x \in Kh^{0,3}(K')$. Suppose that x dies at the nth page of the spectral sequence. Then we have either $d_n(x) = 0$ or $d_n(y) = x$ for some y. But since $Kh^{i,j}(K')$ have no generators in gradings $(1, 4n+3)$ or $(-1, -4n+3)$ for any $n \geq 1$ we conclude that x must survive to the E^∞ page. Hence $s_{max}(K') = 3$ and $s(K') = 2$.

The we conclude using the corollary, that the Conway knot is not slice.

3.2 Slice-Bennequin inequality

The slice-Bennequin inequality relates the braid representation of a link to its Euler characteristic.

Theorem 3.1 (Slice-Bennequin inequality). *Let L be a link with μ components that is represented as the closure of an n-strand braid β and let $w(\beta)$ denote the writhe of its diagram. Let $\chi_S(L)$ denote the maximum Euler characteristic of any smooth properly embedded oriented surface in B^4 with boundary L.*

$$\chi_S(L) \leq n - w(\beta)$$

This theorem was originally proved by Lee Rudolph using Gauge theoretical methods. In this section we use the Rasmussen invariant to give a purely combinatorial proof of this inequality, following [Shu04].

Lemma 3.2. *Let β and β' be two braids such that $\beta = w_1w_2$ and $\beta' = w_1\sigma_i^{\varepsilon}w_2$ where w_i are some braid words and $\varepsilon \in \{\pm 1\}$. Then*

$$|\chi_S(\beta) - \chi_S(\beta')| \leq 1$$

Proof. If $\bar{\beta}$ is a knot, then since $\chi_S(\beta) = 1 - g_*(\bar{\beta})$ we have

$$g_*(\bar{\beta}) \geq \frac{w(\beta) - k + 1}{2}$$

Now suppose that $\bar{\beta}$ is a link. Denote by β_+ the braid obtaining by removing from the braid representation all the standard generators with negative exponents. Since cancelling a pair of generators $\sigma_i\sigma_i^{-1}$ doesn't change the writhe or the maximal Euler characteristic, we can always suppose without lost of generality that the closure of β_+ is a knot, moreover a positive knot and the equality

$$\chi_S(\beta_+) = O(\beta_+) - w(\beta_+)$$

holds for it. If one adds a negative crossing to a braid, this decreases the writhe by 1 so the right hand side increases by 1. The previous lemma says that the left hand side cannot change by more than 1. This concludes the proof. □

3.3 Producing exotic $\mathbb{R}^4$'s

In this section we show the existence of a family of topologically slice but not smoothly slice knots, using the Rasmussen invariant and an exercice (and solution) given in [GS99].

Definition 3.2. *A knot K is said to be topologically slice (sometimes called topologically locally flatly slice) if K bounds a disk D which is properly embedded in B^4 such that locally the pair (D, B^4) is homeomorphic to the pair $(\mathbb{R}^2, \mathbb{R}^4)$. K is said to be smoothly slice if it bounds a disk which is smoothly embedded in B^4 ie. if the pair (D, B^4) is locally diffeomorhic to (R^2, R^4).*

Until now, our definition of being slice was that of smoothly slice. Smooth sliceness implies thetopological sliceness, but the inverse is not always true. Moreover such knots provide examples of exotic structures on $\mathbb{R}^4$ as the following proposition suggests

Proposition 3.1. *Let K be a knot which is topologically slice but not smoothly slice. Then the trace $X(K)$ of the knot embeds smoothly in some exotic $\mathbb{R}^4$ R ie. R is homeomorphic to $\mathbb{R}^4$ but not diffeomorphic.*

We will make use of the following important result

Theorem 3.2. *Any connected, non compact* 4*-manifold admits a smooth structure.*

Proof. (of 3.1) Let K be such a knot. Let ϕ be a topological embedding, and note that $\mathbb{R}^4 \setminus int(\phi(X(K))$ is connected and non compact so by the lemma it admits a smooth structure. By uniqueness of smoothings on 3-manifolds, the homeomorphism ϕ restricted on $\partial(X(K))$ is isotopic to a diffeomorphism, so the smooth structures on $X(K)$ and of $\mathbb{R}^4 \setminus int(\phi(X(K))$ fit together to give a smooth structure R on $\mathbb{R}^4$. Since K is not smoothly slice, by the trace lemma $X(K)$ cannot smoothly embed into $\mathbb{R}^4$, so R should be an exotic $\mathbb{R}^4$. □

Now let's find a family of such knots computing their Rasmussen invariant. Again, we follow [Shu04].

To detect the topological sliceness, we use the following theorem due to Freedman and Quinn in [FQ90].

Theorem 3.3. *If* $\Delta_K(t) = 1$ *then* K *is topologically slice.*

Recall that in Chapter 2)we get a condition on the filtration spectral sequence under which Bar Natan's conjecture is true. It is that the spectral sequence converges after the E_4 term. Indeed more generally we have

Lemma 3.3. *Let* K *be a knot such that* $rank(Kh^{i,j}(K))rank(Kh^{i+1,j+4n)}(K)) = 0 \forall i, j$ *and* $n \geq 2$*. Then Bar Natan's conjecture holds for* K*.*

Proof. Indeed since the differential d_n of the E_n page is of bidegree $(k, 4n)$, the condition on the ranks imply that the d_n is trivial for $n \geq 2$, and the rest of the arguments is the same as in chapter 2. □

Whenever Bar-Natan conjecture holds, one can compute the Rasmussen invariant using only Khovanov homology.

Also, it is possible for a knot to have homological width equal to 4, but still satisfy the conditions of the above lemma. See the table below

	-9	-8	-7	-6	-5	-4	-3	-2	-1	0	1	2	3	4	5	6
9																**1**
7															**2**	
5														**2**	**1**	
3												**2**	**3**	**2**		
1											**2**	**3**	**2**			
-1										**4**	**4**	**3**				
-3									**4**	**5**	**3**					
-5								**3**	**4**	**2**						
-7						**1**	**3**	**4**	**2**							
-9					**1**	**2**	**3**	**1**								
-11					**1**	**3**										
-13			**1**	**1**	**2**											
-15		**1**														
-17		**1**														
-19	**1**															

Figure 3.3: Homology ranks of the knot 16^n_{809057}. The homological width is 4 but d_2 of bidegree $(1, 8)$ is trivial.

Shumakowitch considered all the knots having less than 15 crossings and Alexander polynomial $\Delta_K = 1$. They all have homological width equal to 3, and among knots having 16 crossings, 42 of them satisfy 3.3. In total, there are 82 knots with up to 16 crossings being topologically locally-flatly slice and not smoothly slice, since their Rasmussen invariants were computed using only Khovanov homology data

Number of crossings	11	12	13	14	15	16
Number of non-alternating knots	185	888	5110	27436	168030	1008906
among them with $\Delta = 1$	2	2	15	36	145	499
among them with $s \neq 0$	0	0	1	1	15	65

TABLE 3. Number of non-alternating knots that have Alexander polynomial 1 and those among them that have non-zero Rasmussen invariant.

Figure 3.4

One of these knots is the pretzel knot $(-3, 5, 7)$ noted 15^n_{113775}. It is a strongly quasipositive knot and for strongly quasipositive knots we have a formula to compute the Rasmussen invariant:

Proposition 3.2. *Let K be a knot that can be represented as the closure of a quasipositive braid β with K strands and b bands, that is β is a product of b factors of the form $w\sigma_i w^{-1}$. Then*

$$s(K) = 2g_*(K) = b - k + 1$$

knot K	$s(K)$	$hw(K)$	$e(K)$	$E(K)$	knot K	$s(K)$	$hw(K)$	$e(K)$	$E(K)$
13^n_{1496}	2	3	0	8	14^n_{7708}	2	3	0	8
15^n_{28998}	2	3	-4	8	15^n_{87941}	2	3	-2	8
15^n_{40132}	2	3	0	8	15^n_{89822}	-2	3	-8	0
15^n_{52282}	2	3	0	6	$\mathbf{15^n_{113775}}$	**2**	3	**2**	12
15^n_{54221}	-2	3	-8	2	15^n_{132396}	2	3	0	6
15^n_{58433}	2	3	-2	8	15^n_{139256}	2	3	0	10
15^n_{58501}	2	3	-2	8	15^n_{145981}	-2	3	-8	0
15^n_{65084}	2	3	0	8	15^n_{165398}	2	3	0	6
15^n_{65980}	-2	3	-8	2					
$\mathbf{16^n_{412372}}$	$\mathbf{-2}$	3	-12	$\mathbf{-2}$	$\mathbf{16^n_{955859}}$	**2**	3	**2**	12

TABLE 4. All knots with at most 15 crossings and two knots with 16 crossings that have Alexander polynomial 1 and non-zero Rasmussen invariant. The two 16-crossing knots or their mirror images can potentially be strongly quasipositive.

Figure 3.5

3.4 Rasmussen invariant and the smooth Poincaré conjecture

Let us continue to explore the importance of the Rasmussen invariant and why it can be considered as an exciting invariant. In [Fre+09], a computation of Rasmussen invariant of a certain knot was performed by a computer during 766 hours.
[Fre+09] deals with the situation where the smooth Poincaré conjecture can be reduced to the question of determining whether a certain homotopy 4-ball B_1 is the standard B^4. The computation of the Rasmussen invariant enters the game by the following fact, for the a knot said K_2 in the paper

Fact 1. *If $s(K_2) \neq 0$, then $B_1 \neq B^4$ and the smooth Poincaré conjecture is false.*

Sadly, after 766 hours of computation the Rasmussen invariant was computed to be 0, giving no information. While the computation for another potential knot namely K_3 in [Fre+09] was in progress, it was proved in [Akb09] that the concerned balls are standard. However the authors remain positive about the Rasmussen invariant to be a potential invariant to detect smooth structure. The method of computing $s(K_2)$ was by computing the Khovanov complex. Indeed, as we saw in Chapter 2, when s coincides with the integer in Bar Natan's conjecture, then s can be computed only using Khovanov homology. Indeed it will be given by the formula

$$q^{s(K)} = \frac{P_{Kh}(K)(q, q^{-4})}{q + q^{-1}}$$

3.5 An upper bound for the Rasmussen invariant

Even though some families of knots depend only on the Khovanov homology, this is not always the case and one might has to consider Lee homology or the Khovanov-Lee spectral sequence instead to compute Rasmussen invariant, as we saw an example in the case of Conway knot. Luckily, Lee homology for a knot is pretty simple as it is only supported in homology degree 0. Then one only has to consider the following part of the complex

$$CLee^{-1}(D) \xrightarrow{d_{-1}} CLee^{0}(D) \xrightarrow{d_0} CLee^{1}(D)$$

Moreover we know that $Lee(K)$ is generated by two classes and we know explicit representatives for them which are

$$\mathfrak{s}_o + \mathfrak{s}_{\bar{o}}, \mathfrak{s}_o - \mathfrak{s}_{\bar{o}}$$

Hence to determine s, one only needs to know the map d_{-1}. Recall that for a positive knot, we knew that it was 0, which allowed us to compute s. In [Lob09], Lobb studied this map and obtained a diagram depending upper bound $U(D)$ for $s(K)$, aswell as an error estimate so that

$$U(D) - 2\Delta(D) \leq s(K) \leq U(D)$$

he also gave a family of knots for which $2\Delta(D)$ vanishes, so $s(K) = U(D)$.

3.6 A spectral sequence and Rasmussen invariant of odd 3-pretzel knots

Recall that given a diagram D, choosing a crossing and resolving it in two different ways gives us a short exact sequence

$$0 \to CKh(D_1) \to CKh(D) \to CKh(D_0) \to 0$$

this spectral sequence is one of few theoretical computational tools that we have seen in these notes, and it turns out that the bookkeeping is pretty hard to work with it. In [Tur] Turner defines a spectral sequence using skein diagrams where he resolves m crossings by 0 resolution and n crossings by 1-resolution.

Proposition 3.3. *There is a spectral sequence $(E_r^{*,*}, d_r)$ where d_r is of bidegree $(-r, r-1)$ converging to $Kh^{*,j}(D)$.*

A direct application of this new spectral sequence was to compute Khovanov homology for a family of links, namely $(3, q)$-torus links.
Another application was given by the paper [Suz06], in which Suzuki computes the Rasmussen invariant for a family of knots, namely pretzel knots

Theorem 3.4. *Let p, q odd and r an even positive number. If K is the pretzel knot $P(-p, -q, r)$ then*

$$s(K) = p - q$$

and

Theorem 3.5. *Let p, q, r be odd and bigger than or equal to 3. Let K be the pretzel knot $P(p, -q, -r)$. Then*

$$\begin{aligned} s(K) =& 0 if p \geq \min(q, r) \\ & 2 if p \leq \min(q, r) \end{aligned}$$

3.7 Bar-Natan's reformulation and generalization of the Rasmussen invariant

In [Bar02] and [Bar05], Dror Bar-Natan gave respectively a rich reformulation of the Khovanov homology and a generalization to tangles. Since then, many generalizations and computations of the Rasmussen invariant have been developped.
In [MTV12], authors consider Bar-Natan's reformulation and generalization to generalize the Rasmussen invariant as group homomorphisms from the concordance group to $\mathbb{Z}$. Rasmussen-type invariants were obtained by considering finite fields $\mathbb{F}_p$. In particular it is known that some of these invariants verify linear independance as such homomorphisms, as briefly explained in this cute postcard [LZ21]. Now since we have a generalization of Khovanov homology to tangles, one can wonder if there is a way of computing the invariant s by decomposing a knot to tangles. This point of view was recently investigated by Sano and Kim in [SK25]. They recovered the Rasmussen invariant for the family of pretzel knots given in [Suz06]. There is some hope that this local approach might make

the computations and algorithms easier and faster.
Finally, the reformulation of Bar-Natan allowed to generalize the Rasmussen invariant to links, this was done in [BW06].

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