Braids and torus knots

These are some notes (in progress) for myself on basics of braids and further applications to knot theory. Proofs are to complete. I follow Murasugi's "Knot Theory and Applications".

Definitions

Definition 0.1. *The level plane is the set* $E_s = \{(x, y, z) \mid z = s\}$ *for every real number* z*.*

Definition 0.2. *Let* $\mathbb{D}$ *be the unit cube in* $\mathbb{R}^3$*. Define* n *points* A_i *on top,* B_i *on the bottom*

$$A_i := (\frac{1}{2}, \frac{i}{n+1}, 1)$$
$$B_i := (\frac{1}{2}, \frac{i}{n+1}, 0)$$

Draw n *arcs* d_i *to* $\mathbb{D}$ *such that*

- d_i *are mutually disjoint*
- d_i *begins at* A_i*, ends at* B_k
- $\forall s$ *and* $i, E_s \cap d_i$ *is a point*
- $\forall i, d_i \subset \mathbb{D}$

The resulting collection of n *arcs* d_i *is called a* n*-braid or a braid with* n *strands. We denote* B_n *the set of all* n*-braids.*

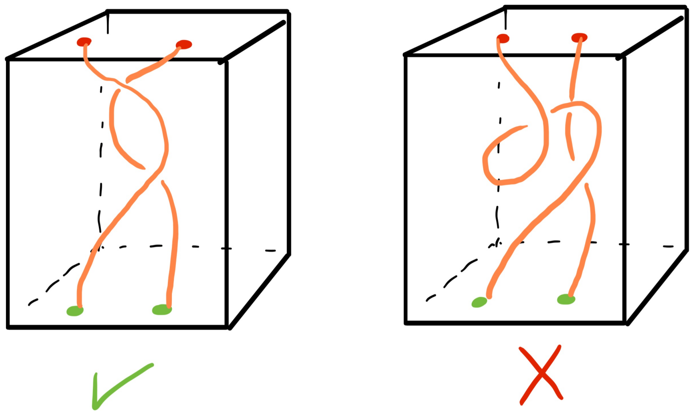

Figure 1: Example and counterexample of 2-braid

Example 0.1.

Definition 0.3. *We say that two braids are equivalent if we can transform one to the other using elementary knot moves. As in knots, we can obtain the regular diagram of a braid by projecting the braid onto the yz-plane*

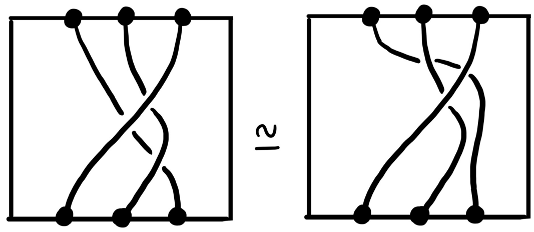

Figure 2: Equivalent braids

Example 0.2. *The trivial braid is obtained by joining A_i to B_i for all i*

The braid group

To a braid, we can assign a permutation. Suppose 1 connects to i_1, 2 connects to i_2.. etc.

If two braids are equivalent, in particular their braid permutations are the same. Hence it gives us a braid invariant. Now consider $B_n/$ where is for the equivalence relationship.

Definition 0.4. *We can define the product of two braids as gluing one atop the other, that is we glue the base of one cube on the top face of the other. It is a vertical juxtaposition.*

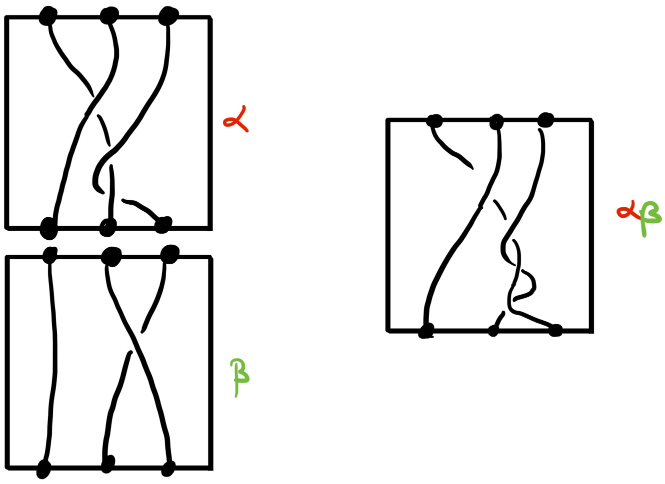

Figure 3: Braid product

Remark 0.1. *In general we do not have $\alpha\beta = \beta\alpha$! One can verify that under this operation the unit is the trivial n-braid and the inverse of a braid B is the mirror image of B denoted B^{-1}.*

Definition 0.5. *Among the n-braids, we can create certain specific n-braids connecting A_i to A_{i+1} and A_j to A_k with $k \neq i, i+1$ by line segments as below. In this way, we can create $n-1$ n-braids.*

We denote these type of n-braids by σ_i

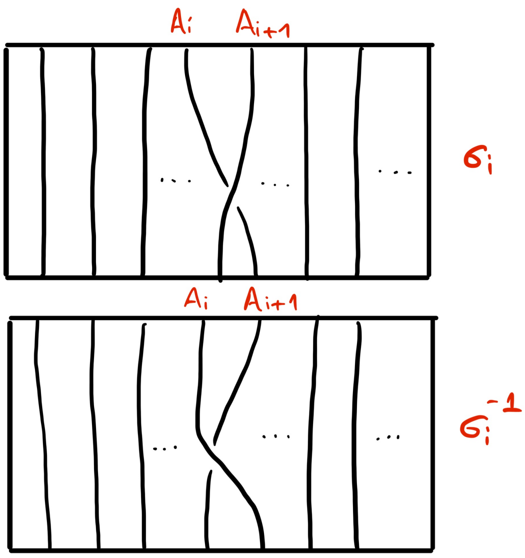

Figure 4: Elemntary n-braids

Braid relations

It is not hard to see that they generate the braid group. To see this, take a braid diagram and divide it into horizontal segmenrs so that in each rectangle there is only one crossing. By definition of the braid product, we can decompose the braid into a product of elementary braids.

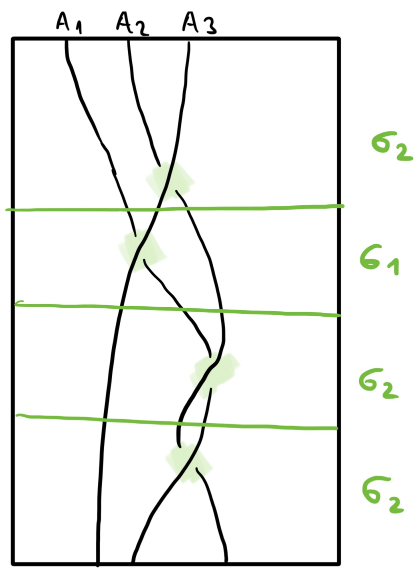

Figure 5: Decomposition

Now, we have an algebraic way of describing a braid. However notice that these algebraic representations are not unique! Indeed we allow crossings to change levels. For example in B_4 we have

$$\sigma_3\sigma_1 = \sigma_1\sigma_3$$
$$\sigma_1\sigma_2\sigma_1 = \sigma_2\sigma_1\sigma_2$$

These type of relations are called the braid relations in B_n. If two braid representations are equiavalent, we can change one to the other using these relations. Below is a fundamental result on the braid group B_n

Theorem 0.1. *B_n has only two types of relations, called fundamental relations:*

(1) $\sigma_i\sigma_j = \sigma_j\sigma_i, |i-j| \geq 2$

(2) $\sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1}$

Hence the braid group is defined as the group generated by $n-1$ elements and the fundamental relations.

Knots and braids

Definition 0.6. *Let us connect, by a set of parallel arcs with endpoints A_i' that lie outside the square, the points A_i to A_i' as below*

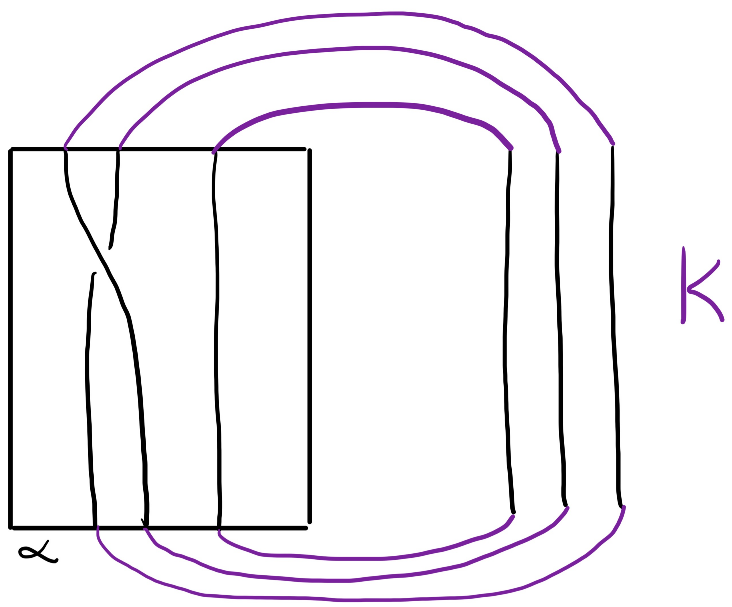

Figure 6

So from a braid, we obtain a regular link diagram. We say that the link K is created from the braid α or that K is the braid closure of α. We can always orient strings from A_i to A_i' so that the we obtain an oriented link.

Below is a fundamental result

Theorem 0.2. *Alexander's theorem Given an arbitrary (oriented) knot (or a link), it is equivalent to a knot (or a link) that is obtained from a braid.*

Proof. □

If two braids are equiavlent, it is clear that their knots are equivalent, since the obtained diagrams are related by knot moves. But it is also possible to obtain equivalent knots from non equivalent braids. Below is an example of the unknot obtained from non equivalent braids

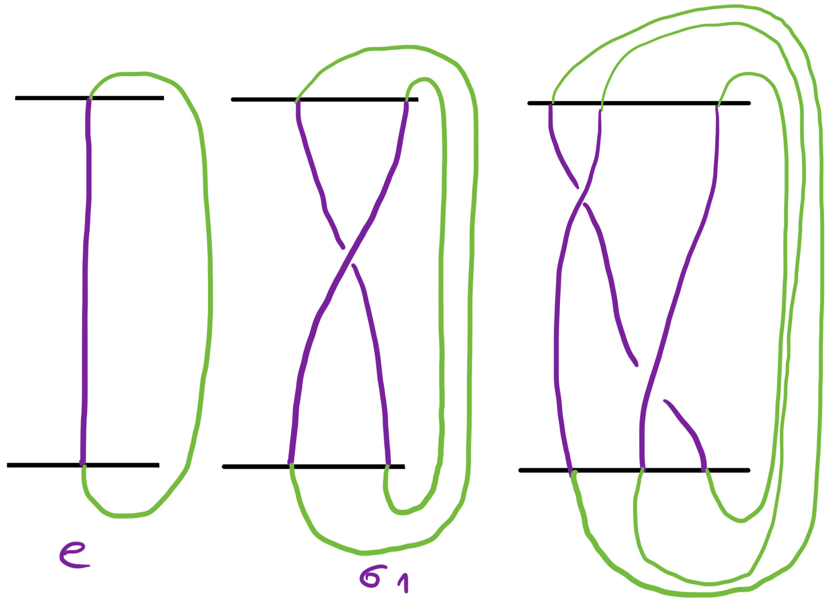

Figure 7: Unknot as different braid closures

Markov moves

If we want to apply the braid theory to knots, we may want to consider together all the braids which give equivalent knots. Hence we consider the concept of M-equivalence of two braids.

Definition 0.7. *Let $B_\infty := \bigcup_{i\geq 1} B_i$. On B_∞ we want to perform the two following operations, known as Markov moves:*

(1) If b is ab element of B_n then

$$M_1(b) = \gamma b \gamma^{-1}, \gamma \in B_n$$

ie. the conjugate operation

(2) If $b \in B_n$,

$$M_2(b) = b\sigma_n \text{ or } b\sigma_n^{-1}$$

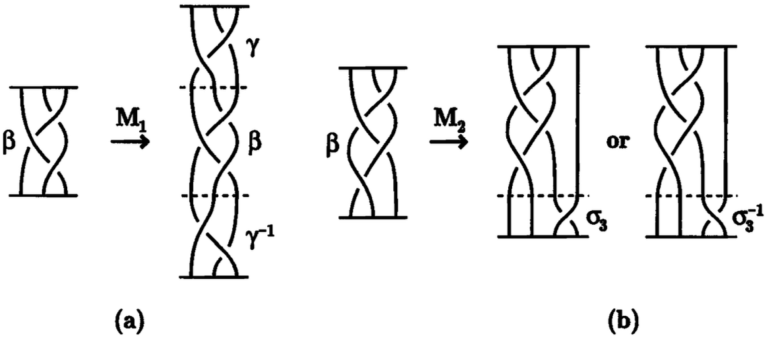

Figure 8: Markov moves

Definition 0.8. *Let $\alpha, \beta \in B_\infty$. If we can transform α to β using M_i, M_i^{-1} a finite number of times, then we say that two braids are Markov equivalent.*

Theorem 0.3. *Markov's theorem Let K_1, K_2 be two oriented knots (or links) which can be formed by b_1, b_2. Then*

$$K_1 \cong K_2 \leftrightarrow b_1 \cong_M b_2$$

The braid index

Then a knot K can be formed from an infinite number of braids. Within this set of braids which give rise to K, there exists a braid having the minimum number of strands. This number is called the braid index $b(K)$ of the knot K. For example the only knot with braid index 1 is the unknot. The knots with braid index 2 are the torus knots of type $(n, 2)$ with $n \neq *, \pm 1$. The links with braid index 3 are difficult to list. Note that the braid index depends on the orientation. In same cases, as in the case of torus knots, we can completely determine the braid index.

Torus knots

Consider the following diagram for the torus knot $T_{q,r}$

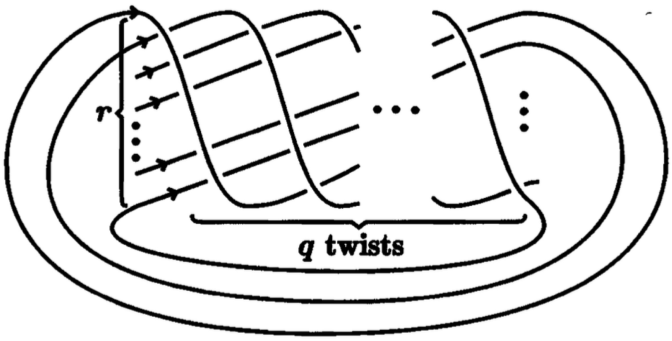

Figure 9

It is clear from this diagram that we can obtain $T_{q,r}$ as the closure of the r braid

$$(\sigma_{r-1}\sigma_{r-2}\cdots\sigma_2\sigma_1)^q$$

The bridge number

At each crossing point of a regular diagram D of a knot K, we remove a fairly small segment AB that passes over the crossing point. The result of removing these segments is a collection of disconnected arcs. We may think of the original regular diagram D as the resylting diagrams that occurs when we attach segmenr AB that pass over to the end points of these disconnected polygonal curves on the plane

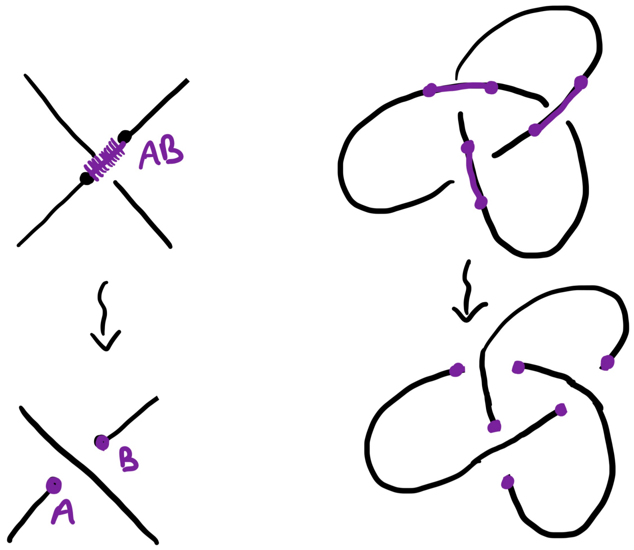

Figure 10

These segments are called bridges. If D is a link diagram, the given number of bridges is called the bridge number of D. It is not a link invariant, since the 1- crossing and the 0-crossing diagrams of the unknot have respectively bridge numbers 1 and 0.

Definition 0.9. *Let D be a regular link diagram of a link. Divide D into $2n$ polygonal curves: α_i, β_i with*

$$D = \alpha_1 \cup \alpha_2 \cup \cdots \cup \beta_1 \cup \cdots \beta_n$$

such that

(1) $\alpha_1, \cdots, \alpha_n$ are mutually disjoint simple polygonal curves

(2) $\beta_1, \cdots, \beta_n$ are also mutually disjoint simple polygonal curves

(3) At the crossing points of D, α_i are on segments that pass <u>over</u> the crossing points, where β_i are on the segments passing <u>under</u>.

Take the minimal n such that

$$br(D) \leq n$$

and call it the bridge number $br(D)$ of the diagram. The example below shows that it is not a link invariant

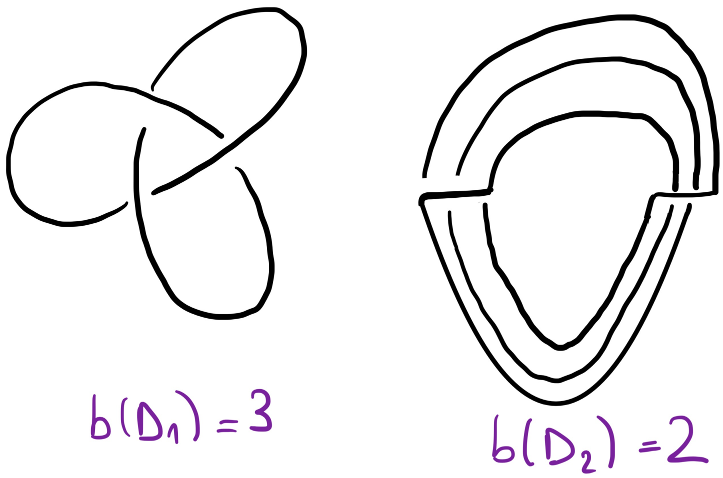

Figure 11: two different diagrams of the trefoil have different bridge numbers

Clearly, the minimum bridge number among all the diagrams is a link invariants.

Theorem 0.4. *For a link L,*

$$br(L) = \min\{br(D) \mid D\}$$

is a link invariant.

Theorem 0.5.

$$br(K_1 \# K_2) = br(K_1) + br(K_2) - 1$$

Conjecture 0.1. *If K is a knot,*

$$c(K) \geq 3(br(K) - 1)$$

where the equality holds only for the trivial knot, trefoil knot or the square/granny knot.

It is possible to compute the bridge number in a different way then that described above. For this, we suppose that the regular diagram is smooth.
Denote by $\vec{v}(D)$ the number of local maxima of Di in relation to the direction of a certain plane vector $\vec{v}$ of the plane as below

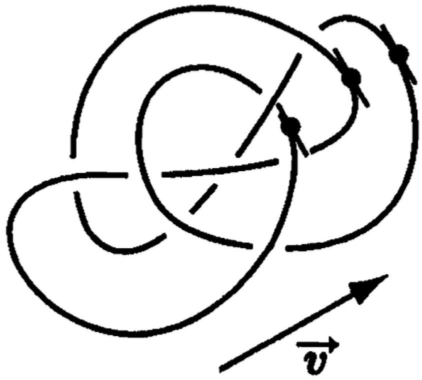

Figure 12

It is clear that $\vec{v}(D)$ is not a link invariant, all the Reidemeister moves increase the number, but if we consider all the link diagrams and take the minimum value over, we will get a link invariant. This knot invariant is also equal to $br(K)$ by the following theorem

Theorem 0.6.

$$br(K) = \min_D \vec{v}(D)$$

Bridge number and the braid index

Let K be a knot, α the minimum braid presentation of K. Then $b(K)$ is the number of strands of α. Since K is obtained joining $b(K)$ parallel strands, that diagram of K has $b(K)$ local maximas hence we easily have

$$br(K) \leq b(K)$$

In general determining the braid index is a difficult problem. Let us see what happens in the case of torus knots. We already have

$$br(T_{q,r} = \min(q, r)$$

and since $T_{q,r} = T_{r,q}$, assume $r < q$. Since

$$T_{q,r} = c((\sigma_{r-1} \cdots \sigma_1)^q)$$

we have

$$b(T_{q,r}) \leq r = br(T_{q,r})$$

so

$$b(T_{q,r}) = br(T_{q,r}) = \min\{q, r\}$$